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ABSTRACT

This eleventh unit in the SMSG secondary school mathematics series is the teacher's commentary for Unit 9. First a general overview of the entire FIRST COURSE IN ALGEBRA (Units 9 and 10) is provided. Then, a time allotment for each of the chapters in Unit 9 is suggested. For each of the chapters in Unit 9, the goals for that chapter are discussed, the mathematics is explained, some teaching suggestions are provided, the answers to exercises are listed, and sample test questions for that chapter are suggested. (DT)

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SMSG

SE 021 989

School Mathematics Study Group

First Course in Algebra

Unit 11

First Course in Algebra

Teacher's Commentary, Part I

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PREFACE

The principal objective of this FIRST COURSE IN ALGEBRA is to help the student develop an understanding and appreciation of some of the algebraic structure exhibited by the real number system, and the use of this structure as a basis for the techniques of algebra. More specifically, we are interested in an exploration of the properties of addition and multiplication of real numbers and their order properties. Later in the course we have occasion to consider polynomials, which, as a class, also exhibit an algebraic structure which derives from the fact that they satisfy some (but not all) of the above set of properties.

The teacher is urged to be on the watch for these properties as they appear and reappear through the course, and to keep in mind that they form the basis for all of the algebraic structure which we hope to discover in the real number system.

The principal objective of this COMMENTARY FOR TEACHERS is to give all possible aid to the teacher as he leads the student toward the above objective. Just as we urge the student to read his text carefully, we also urge the teacher to make full use of this commentary.

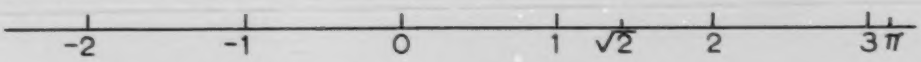
To help the teacher and student alike in this regard the textbook has been written in a special way. First, it is in "spiral" form, so that virtually every concept introduced appears repeatedly later, broader and/or deeper in meaning with each recurrence. The spiral form suggests, incidentally, that the teacher need not expect of his class complete mastery of all details of a topic before moving ahead in the text, for he will find opportunity to help his class polish the rough edges as the topic comes up again. His principal concern may well be rather that the student is confident enough to continue without uneasiness, and this may offer the clue as to the pacing of the course.

Second, text and exercises are interrelated in a complementary way, so that, for example, exercises often carry forward the development of ideas presented earlier in the text and also hint at new ideas lying ahead. It follows from the above that the student will find it necessary to read the text regularly and with considerable understanding. This, we feel, is a reasonable expectation for the students for whom this course is intended--the "college-capable" (not necessarily college-bound) or upper one-half to one-third, academically, of the school population.

Perhaps the most desirable way for someone with considerable mathematical maturity to study any algebraic system is to begin with a set of axioms which describe the system and then systematically to develop the system by formulating definitions, proving theorems, constructing examples, etc. to whatever extent the ability of the individual allows. This is the purely deductive approach. The difficulty with this approach is that the student must possess enough knowledge and experience so that he either already has or can construct (perhaps with some help) at least a tentative model of the axiom system. Without this, work with the axioms degenerates into just so much formalism. The student of Euclidean geometry may have more success with a purely deductive approach, because he already has a highly developed geometric intuition obtained from living in a three-dimensional space. It is rather obvious, however, that the ninth grade student's intuition concerning the real numbers is far from adequate to support an axiomatic approach. For this reason, we intend to be quite informal and intuitive, but *not* incorrect.

It is important to state as explicitly as we can just what is assumed concerning the student's knowledge of the real number system when he enters ninth grade. The following assumptions appear to fit the facts closely enough to provide us a starting point.

Assumptions: The student is more or less familiar with the *class* of objects which we call the real numbers. This includes the negative reals (via experiences such as those involving gains and losses, degrees below zero on a thermometer, etc.) as well as certain irrationals such as $\sqrt{2}$ or π. The "picture" which goes with the class of real numbers is the full number line,



in which it is taken for granted that there is a number for each point and conversely. Notice that the order relation is implicit in this picture, although the operations of addition and multiplication are virtually absent.

The student probably has some vague notions concerning algebraic structure from his training in arithmetic. This, of course, involves primarily the non-negative numbers. However, the student also has some notion of adding negatives and multiplying negatives by a positive (adding losses, losing a certain amount each day for a given number of days, etc.)

Some students who have studied the School Mathematics Study Group material for grade 7 or grade 8 or both will know more about real numbers and algebraic structure than has been described above. Such students will be able to move more quickly through some parts of this course. They will not, however, find any of the material sufficiently repetitive to warrant omitting it.

With the assumptions stated above as a starting point, it becomes clear that two things need to be done:

(1) Extend the operations of addition and multiplication to the entire class of real numbers. (It is necessary to emphasize again our point of view that we *already have* the class of *all* real numbers and that we already have the operations of addition and multiplication defined for non-negative real numbers, since these numbers are nothing more or less than the so-called "numbers of arithmetic." This point of view avoids many of the difficulties usually encountered in "introducing" the negatives, etc.)

(2) Make a careful study of the real number system in order to bring out its algebraic structure. The guide here is always the use of the basic properties of addition and multiplication and of order. It is necessary to reexamine the operations of arithmetic so as to prepare the way for the "discovery" that each of the properties is true for the real numbers. After a few properties have been obtained, we can then prove a few simple theorems and so gradually work up to the deductive approach.

It turns out to be desirable to do a part of (2) before (1), since a knowledge of the properties which addition and multiplication ought to have makes it considerably easier to motivate the extension of these operations to the negative numbers. Thus we choose to begin with a careful study of the algebraic structure of the *non-negative real numbers*, drawing heavily on previous knowledge of arithmetic. In this way we are able to discover and state precisely the usual commutative, associative, and distributive properties for addition and multiplication as well as properties of 0 and 1, before becoming involved with the arithmetic of negative numbers.

Thus far we have been speaking of algebraic structure. We must not lose sight of the fact that manipulative facility, while not an end in itself, _is_ important. A multitude of exercises is absolutely necessary for gaining the needed techniques with algebraic symbolism, but these techniques must be tied to the ideas from which they derive their validity. In this way, the student will not have to unlearn either his concepts or his attitudes in future years.

We find that it _is_ possible to relate the necessary techniques to the ideas of our number system about which we are building the course. We propose to keep this relation between the techniques and the ideas a close one.

An outline of the high points of the course now follows.

1. We begin with an informal introduction to sets of numbers and manipulations among them. The examples exhibit, at the same time, some interesting and perhaps surprising properties of sets of numbers of arithmetic. Another view of these numbers is obtained by associating them with points on a line. This device of "picturing" sets of numbers is one which we use to illustrate or motivate many concepts through the course.

2. The student next is invited to recall the many different names (numerals) for a number, and he learns to write sentences to express the fact that two numerals name the same number. Then from his experiences with numbers are extracted the associative and commutative properties of addition and multiplication and the distributive property connecting the two operations. Also, he extracts the properties of 0 and 1 as the respective identities for these operations. Notice that, at this stage, these properties are considered for the non-negative numbers only. At this point, variables are introduced as numerals to represent unspecified numbers.

3. The language of sentences--equalities, inequalities, compound sentences--is learned, and open sentences are defined. The truth set of an open sentence, i.e., the set of values of the

variable for which the sentence is true, and the graph of the truth set are introduced. Then the basic properties of numbers are stated formally in terms of open sentences.

4. The student learns to translate back and forth between English and algebra, and thus begins to be at home with "word problems", which include inequalities as well as equations.

5. We now look at the negative half of the number line. We give names to the points to the left of zero, and define opposites, absolute value, and order relations. The set of numbers assigned to points on the entire number line is called the set of real numbers.

6. Addition for the real numbers is motivated through the following principles: It must be consistent with the familiar addition for the positive numbers, it must continue to satisfy the relevant laws under 2. above, and it must agree with our intuition on addition. We see that $a + (-a) = 0$, and we prove our first theorem: If $x + y = 0$, then $y = -x$. We like the idea of having theorems in the first algebra course, but we certainly expect the emphasis on them to vary from class to class.

7. Multiplication is introduced in accordance with the same principles which were used for addition. If the distributive property is to continue to hold, then the product of two negative numbers must be positive; this is the soundest motivation we know for the definition of multiplication of real numbers.

8. The relation $a < b$ was defined earlier on the number line for positive numbers and extended in 5. to all real numbers. The comparison and transitive properties were stated for order. Now the addition property of order is asserted, and various consequences of this basic property are proved: $a < b$ if and only if there exists a positive number c such that $a + c = b$; if $a < b$ and $0 < c$, then $ac < bc$; if $a < b$ and $c < 0$, then $bc < ac$. At this point, the basic properties of the operations on real numbers are summarized. The structure of algebra rests on this foundation of properties.

9. Subtraction and division are not new operations with independent properties, but are defined in terms of adding the opposite, and multiplying by the reciprocal. There is, by the

way, no separate section on fractions. They are practiced a great deal throughout, with a concentrated dose of exercises at this point when the machinery is complete.

10. A careful definition of a "factor" requires you to state what you are willing to accept as a factor - i.e. "over what" are you factoring. We first factor positive integers over positive integers, and prove some theorems about factors. Simple laws of exponents fit here naturally.

11. The student is now ready for a definition of "root" and a proof that $\sqrt{2}$ is not a rational number. Then further work with radicals leads to the familiar operations on radicals. We use the iteration method for extracting square roots.

12. We can factor polynomials into polynomials with either integer or arbitrary real coefficients; the possibility of non-real coefficients must be left open. A polynomial is thought of as the result of repeatedly applying addition and multiplication to real numbers and a specified number of variables. Factoring of polynomials bears a great resemblance to the previous factoring of integers and leads to solutions of certain quadratic equations.

13. The solution of equations and inequalities in one variable is based on finding ways of transforming such equations and inequalities, by operations which leave solution sets invariant, into equations or inequalities whose solution sets are obvious. This process was begun for equations in 7. and is now extended to all sentences. When an operation does not yield an equivalent sentence, then we must permit solution sets to increase, and "extraneous" solutions may result. The absolute value provides some fine examples here.

14. Coordinates and graphs are introduced,

15. and applied in finding solution sets of equations and inequalities in two variables in the same spirit as above.

16. A coordinate system is used to study quadratic polynomials. Then quadratic equations are solved when the quadratic polynomial can be factored over the real numbers.

17. There exist many ways of defining an association from one set of numbers to another. For example, a graph, a table of pairs, a formula, a verbal description all give you such an association.

It is the association which is important, not the terms in which it is defined, and such an association is what we mean by a function. Thus, the idea of function unifies all the ideas of operations, correspondences, equations, and algebraic expressions.

We have tried to give a brief picture of the nature of this course. We would now like to make a few comments about the mechanics of using the textbook and the teacher's commentary.

The preface to the student text is intended to prepare the student for the need for careful reading of his book. You should urge your students to read their preface, and from time to time through the year, mention this matter of reading for understanding and how to improve one's reading ability.

The problem sets are not of uniform length, so it is not expected that a teacher can assign exactly one problem set each day for homework. There are many differences in ability of classes, in type of material, in available time, in patterns of assignments, etc. Some teachers prefer to assign each day a few problems from each of two or three successive problem sets. You should use your judgment in selecting problems for assignments. Be sure to include those problems which are essential to the development of the ideas, and whatever additional problems are appropriate to the ability of your students.

In many of the problem sets throughout the book there are included problems preceded by an asterisk *. These are more challenging than others in the same problem set and are included primarily for the brighter and more curious student. The use of these problems with an entire class may consume time needed to complete the basic work of the course. The teacher will have to decide as he reaches each such problem whether time and the ability of his students permit him to deal with the problem with the class as a whole.

This Commentary for Teachers is intended to be more than an answer book. You will find in it discussions of what we are doing, why we choose to present a particular topic in a certain manner, and what comes later to which this topic is leading.

At the ends of some of the chapters in the Commentary there are lists of suggested test items. It should be clearly understood that these lists are not tests, but merely a few sample problems from which you may choose questions or which may suggest to you other questions for your tests.

It is very difficult to suggest how many days a class should spend on any particular chapter. Ability of the students varies and local conditions in the school vary greatly. There are records of different classes spending from 8 to 32 days on the same chapter. Perhaps the best advice is this: try to complete this first part, Chapters 1 to 6 inclusive, by the first third of the school year. Effective use of this Commentary may help you to avoid spending more time than you should on any one topic.

For teachers who want a very rough guess at the number of days to be spent on each chapter, the following estimates are given. Variations from these estimates should be expected.

Chapter 1	6 days
Chapter 2	8 days
Chapter 3	18 days
Chapter 4	10 days
Chapter 5	10 days
Chapter 6	10 days

There is available to the teacher a book written specifically to explain the mathematical thinking behind this course. This useful reference is Haag, *Studies in Mathematics*, *Volume III*, *Structure of Elementary Algebra*. Chapter and section references to this book are made frequently in the commentary which follows.

Chapter 1

SETS AND THE NUMBER LINE

In this chapter we use the numbers of arithmetic and the basic operations upon them as a familiar background for the introduction of concepts and procedures which may be new to the pupil. We consider briefly two of the indispensable tools for our study of the structure of the real number system, sets and the number line.

One of the great unifying and simplifying concepts of all mathematics, the idea of set is of importance throughout the course in many ways--in classifying the numbers with which we work, in examining the properties of the operations upon these numbers, in solving equations and inequalities, in factoring polynomials, in the study of functions, etc. Also, the tenth grade course and later SMSG courses use the ideas of sets in formulating definitions, postulates, and theorems.

Since most students have not studied about sets before entering this course, and since the basic notions of set are usually grasped quite readily, sometimes even eagerly, it seems a good topic, from a motivational standpoint, with which to start the course. We move on quickly from this first discussion of sets, however, postponing much work with operations on members of sets and with closure, so as to get quickly to the presentation of variable (in Chapter 2). This is done largely because (1) teachers and students alike expect the early introduction of variable in an algebra course and (2) with the idea of variable at hand, our study of the structure of the number system can begin in earnest.

Next we place the number line before the student. Here again is a concept that is of use throughout the course; it is the device for picturing many of the ideas about numbers and the operations on them. This is immediately apparent as the graphing of sets is introduced, and is followed, in the final section of the chapter, by addition and multiplication on the number line.

Addition and multiplication on the number line are not, however, brought in for their own sake, since Chapters 6 and 7 deal thoroughly with these operations; rather this section is intended to lay the groundwork, in several ways, for the commutative properties of addition and multiplication. First, it makes clear to the student that, for example, 3 + 5 and 5 + 3 are different symbols; that though both name the number eight, the eight is found differently in each case because addition is an ordered operation. Thus there is meaning in the commutative property of addition. In the same manner this section prepares the student for the commutative property of multiplication. Number line addition and multiplication also illustrate the commutative properties in examples, such as those involving rational numbers in which the fact that the properties hold is not altogether obvious.

The teacher is referred to Haag, Studies in Mathematics, Volume III, Structure of Elementary Algebra, Chapter 2, Section 1.

Pupils who have studied SMSG Mathematics for Junior High School will have had a little experience with sets and the number line. They may be able to go through parts of this chapter a little more quickly than other students, but the treatment is sufficiently different to suggest that none of the ideas should be omitted.

1-1. Sets and Subsets.

Page 1. Although the first sets listed at the outset of the chapter are not examples of sets of numbers, we move quickly in the text to consideration of such sets. In spite of the fact that non-numerical sets may be of interest, a prolonged discussion of them would constitute a diversion, for which time is not likely to be available, from the basic purpose of the course.

Alabama, Arkansas, Alaska, and Arizona are all the states whose names begin with the letter, A.

Monday, Tuesday, Wednesday, Thursday, Friday are all the "school days" of the week. Another description is "all the weekdays", although some might include Saturday in the "weekdays".

All seven comprise all the days of the week.

The numbers 1, 2, 3, 4, 5 are the first five counting numbers, or the whole numbers greater than 0 and less than 6.

The numbers 2, 3, 5, 7, 8 are a set of whole numbers less than 9; this, however is not an adequate description, since it describes other sets as well as the one given. In contrast to the preceding

[page 1]

set, {1, 2, 3, 4, 5}, the set {2, 3, 5, 7, 8} is not easily described in words, but only by listing the elements. At the other extreme is the null set, which cannot be listed and must be described in some other manner.

We do not introduce much of the standard set notation, such as set builder notation, $\in$, $\supset$, $\subset$, $\cup$, $\cap$, because we found that there was not enough use for them to make it worthwhile. There is, however, no objection to the teacher using any of these if he so desires.

Page 2. Though we define and distinguish between the set of whole numbers and the set of counting numbers or natural numbers, no unusual emphasis should be placed upon the latter set, for we find more use for the set of whole numbers. With the discussion of real numbers in Chapter 5 the term *integers* will be introduced to designate whole numbers and their opposites (negatives).

The teacher should be aware of three common errors made by students in working with the empty set. The most important error is the confusion of {0} and ∅, which is warned against in the text. This point, however, may need further emphasis by the teacher. A less significant mistake is to use the words "*an* empty set" or "*a* null set" instead of "*the* empty set". There is but one empty set, though it has many descriptions. A third error is the use of the symbol, {∅}, instead of just ∅.

Incidentally, the symbol used for the empty set is the same as that used as a vowel in the Danish alphabet.

Page 3. T is a subset of S.

Answers to Problem Set 1-1a; Pages 3-4:

1. (a) {1, 3, 5, 7, 9, 11}
   (b) {0, 1, 4, 9, 16, 25, 36, 49}
   (c) {12, 24, 36, 48}
   (d) {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
   (e) This question should be checked annually with the World Almanac.
   (f) {0, 1, 4, 9}
   (g) {0, 1, 4, 9, 16, 25, 36, 49, 64, 81}
   (h) The null set--cannot be listed

2. (a) The elements of each set are the same: 1, 2, 3, 4, 5, 6
   (b) The set S = {1, 2, 3, 4, 5, 6} is the set of the first six counting numbers. Many other descriptions are possible.
   (c) The same set may have many descriptions.
3. U = {1, 2, 3, 4}
   T = {1, 4, 9, 16}
   V = {1, 4}; yes, V is a subset of U; yes, V is a subset of T; no, U is not a subset of T, since 2 is not an element of T
4. K = {1, 2, 3, 4, 9, 16}
   K is not a subset of U; U is a subset of K; T is a subset of K; U is a subset of U (by definition of subset).

Problems preceded by the asterisk * are more challenging than others in the same set of exercises. Such problems are included primarily for the brighter and more curious student, and the use of these problems with an entire class may consume time needed later in the year to complete the basic work of the course. The teacher will have to decide as he reaches each such problem whether time and the ability of his students permit him to deal with the problem with the class as a whole.

*5. From the set ∅ one subset, ∅, can be found.
   From the set A = {0}, two subsets, {0} and ∅, can be found.
   From the set B = {0, 1}, four subsets can be found: {0}, {1}, {0, 1}, and ∅.
   From the set C = {0, 1, 2}, eight subsets can be found: {0}, {1}, {2}, {0, 1}, {0, 2}, {1, 2}, {0, 1, 2} and ∅.
   From the set D = {0, 1, 2, 3}, 2^4 or 16 subsets can be found.
   The student may state the rule in this manner:
   A set with four elements will have twice as many subsets as a set with three elements, and so forth

or

A set with one element has	2	subsets.
A set with two elements has	2 × 2	subsets.
A set with three elements has	2 × 2 × 2	subsets.
A set with four elements has	2 × 2 × 2 × 2	subsets.

etc.

Page 5. There are 50 odd numbers between 0 and 100. They can be counted, but it is not necessary to count them. One way of arriving at the conclusion without counting is to observe that there are 100 whole numbers from 1 to 100, half of which are odd, and half even, and one-half of 100 is 50.

Another way is to realize that there are 5 odd numbers in each decade, and there are 10 decades from 1 to 100, hence there are 50 odd numbers. The pupil might set up a table of these.

The set of all multiples of 5 is infinite in number and cannot be counted with the counting coming to an end.

The use of the term "infinitely many" on the part of the student and teacher should help the student avoid the noun "infinity", and with it the temptation to call "infinity" a numeral for a large number.

Answers to Problem Set 1-1b; Pages 5-6:

1. (a) 18
   (b) 25
   (c) Infinitely many
   (d) 34
   (e) 100
2. (a) Infinite
   (b) Infinite
   (c) Finite
   (d) Finite
   (e) Infinite
3. (a) K = {0}; K is a subset of S; K is a subset of T; S, T, and K are finite.
   (b) M = {0, 2, 4, 5, 6, 7, 8, 9, 10}; M is not a subset of S; T is a subset of M; M is finite.

(c) R = {5, 7, 9}; R is a subset of both S and of M.

(d) A cannot be listed; it is the empty set.

(e) Sets A and K are not the same. A has no elements, while K has one element, 0.

(f) Yes. A subset of a set can have no elements which are not in the set, so the number of elements in a subset is not greater than the number of elements in the set. Any subset of a finite set, therefore, is finite.

4. An even whole number is a whole number which is a multiple of 2, that is, the product of a whole number times 2. 0, being the product of the whole number 0 times 2, is, by this definition, an even number.

1-2. The Number Line.

Page 7. The number line is used as an illustrative and motivational device, and our discussion of it is quite intuitive and informal. As was the case with the preceding section, more questions are raised than can be answered immediately.

Present on the number line implicitly are points corresponding to the negative numbers, as is suggested by the reference to a thermometer scale and by the presence in the illustrations of the left side of the number line. Since, however, the plan of the course is to move directly to the consideration of the properties of the operations on the non-negative numbers, anything more than casual recognition of the existence of the negative numbers at this time would be a distraction to the student.

Page 7. This idea of successor is important. Suppose you begin with the natural number one. The successor is "one more", or 1 + 1, which is 2. So the successor of 5 is 5 + 1, or 6; of 105 is 105 + 1, or 106; and of 100,000,005 is 100,000,006. This implies that whenever you think of a whole number, however large, it always has a successor. To the pupil should come the realization that there is no last number. An interesting reference for the student is Tobias Dantzig, Number, the Language of Science, pp. 61-64.

Page 9. The emphasis here is on the fact that coordinate is the number which is associated with a point on the line. "Coordinate" and "associated" and "corresponding to" must become part of the pupil's vocabulary. He must not confuse coordinate with point, nor coordinate with the name of the number.

The distinction between number and name of a number comes up here for the first time. Do not make an issue of it at this time, for it is dealt with explicitly at the beginning of Chapter 2.

Observe that the general statement on page 9 concerning rational numbers is not a definition, because we do not at this point include the negative rational numbers. Do not make an issue of this with the student. After we have introduced negative numbers in Chapter 5 he will have a definition of rational numbers. For the moment we want him to have the idea that these numbers are among the rationals.

It is also possible to say that a number represented by a fraction indicating the quotient of a whole number by a counting number is a rational number. This statement may be of interest, since it is expressed in terms of these recently-defined sets, but the statement in the text has the advantage that the exclusion of division by zero is explicit.

$\frac{10}{3}$, $\frac{14}{2}$, $\frac{35}{10}$, $\frac{0}{7}$ are possible names for these numbers.

Page 10. A rational number may be represented by a fraction, but some rational numbers may also be represented by other numerals. The number line illustration on page 10 gives the name "2" as well as the fractions "$\frac{4}{2}$", "$\frac{6}{3}$", "$\frac{8}{4}$" to name the number 2. The same diagram makes clear that not all rational numbers are whole numbers. The students may have seen some fractions that do not represent rational numbers, such as, $\frac{\sqrt{2}}{2}$, $\frac{4\pi}{3}$, etc. They will have to be reminded that so-called "decimal fractions" are not, by this definition, fractions.

It is necessary to keep the words "rational number" and "fraction" carefully distinguished. Later on in the course, (Chapter 9) it will be seen that the meaning of the term "fraction" includes any expression, also involving variables, which is in the form of an indicated quotient.

Pages 10-11. The idea of "density" of numbers is being initiated here. By "density" of numbers we mean that between any two numbers there is always another, and hence that between any two numbers there are infinitely many numbers. This suggests that, on the number line, between any two points there is always another point, and, in fact, infinitely many points. We refer here to "points" in the mathematical, rather than physical sense--that is, points **of no dimension**. Because the student may not be thinking of points in this way he may not intuitively feel that between any two points on the number line other points may be located. Therefore, he is shown "betweenness" for numbers first, then, taking these numbers as coordinates, he can infer "betweenness" of points on the number line.

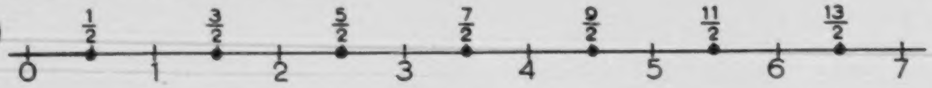
The introduction of other positive numbers besides rational numbers is done quite gradually. An exercise in Chapter 3 suggests that $\sqrt{2}$ might be irrational; in Chapter 5 various radicals and also π are definitely asserted to be irrational, and the careful treatment of the irrationality of various radicals and their sums is given in Chapter 11 after adequate preparation on factoring.

At this point in the course, it is hoped that the student will accept the fact that every point to the right of 0 on the number line can be assigned a number. He may not accept the fact that not every such point has a rational number as its coordinate, but this fact need not be emphasized until Chapter 11. He may also be impatient to assign numbers to points to the left of 0. For the time being, until Chapter 5, we shall concentrate on the non-negative real numbers. This set of numbers, including 0 and all numbers which are coordinates of points to the right of 0, we call the set of numbers of arithmetic. After we establish the properties of operations on these numbers (in Chapter 3) we shall consider the set of all real numbers which includes the negative numbers (in Chapter 5). Then in Chapters 6, 7 and 8 we spell out the properties of operations on all real numbers.

Answers to Problem Set 1-2a; Pages 11-12:

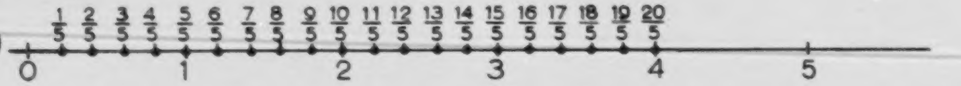
1. Between 2 and 3 there are infinitely many; between $\frac{2}{500}$ and $\frac{3}{500}$ there are infinitely many.

2. (a)



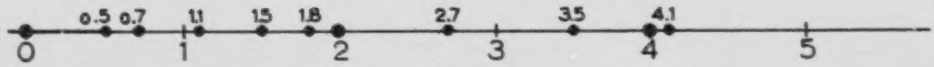
$\frac{1}{2}$
$\frac{3}{2}$
$\frac{5}{2}$
$\frac{7}{2}$
$\frac{9}{2}$
$\frac{11}{2}$
$\frac{13}{2}$
0
1
2
3
4
5
6
7

(b)



$\frac{1}{5}$
$\frac{2}{5}$
$\frac{3}{5}$
$\frac{4}{5}$
$\frac{5}{5}$
$\frac{6}{5}$
$\frac{7}{5}$
$\frac{8}{5}$
$\frac{9}{5}$
$\frac{10}{5}$
$\frac{11}{5}$
$\frac{12}{5}$
$\frac{13}{5}$
$\frac{14}{5}$
$\frac{15}{5}$
$\frac{16}{5}$
$\frac{17}{5}$
$\frac{18}{5}$
$\frac{19}{5}$
$\frac{20}{5}$
0
1
2
3
4
5

(c)



0.5
0.7
1.1
1.5
1.8
2.7
3.5
4.1
0
1
2
3
4
5

Three of these are whole numbers, 0, 2, 4. 2 and 4 are counting numbers.

(d) All the names for numbers in (c) are not fractions. All the names for numbers in (a) and (c) represent rational numbers.

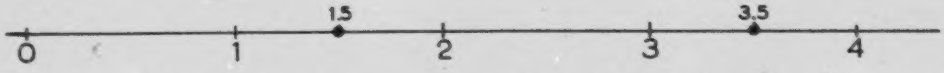
If the student is uncertain how the graph should look, he may refer to the illustrations on page 14 of the text. It is our hope however, that he will not require this assistance, but that instead the exercise will point toward the graphing of sets which follows.

3. $\frac{20}{5}$ is another name for the natural number 4. Other suggested names for the coordinate of this point include $\frac{8}{2}$, $\frac{100}{25}$, $\frac{12.4}{3.1}$, 4.0.

4. Suggested numerals include $\frac{6}{8}$, $\frac{12}{16}$, $\frac{3000}{4000}$, $\frac{1.5}{2}$, $\frac{3.6}{4.8}$, .75.

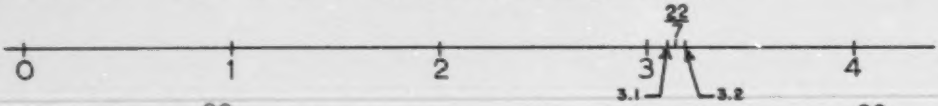
A comparison of the answers to Problem 2 (c) and Problem 4 of these exercises suggests that we do not care whether a decimal between 0 and 1 (or, later, -1 and 0) is preceded by 0. Either form is acceptable.

5.



The point whose coordinate is 3.5 is to the right of the point whose coordinate is 2. The number 3.5 is greater than the number 2. The point whose coordinate is 1.5 is located to the left of the point whose coordinate is 2. The number 2 is greater than the number 1.5. (Note that this problem is leading into work on the order relation of numbers and associated points.)

6.



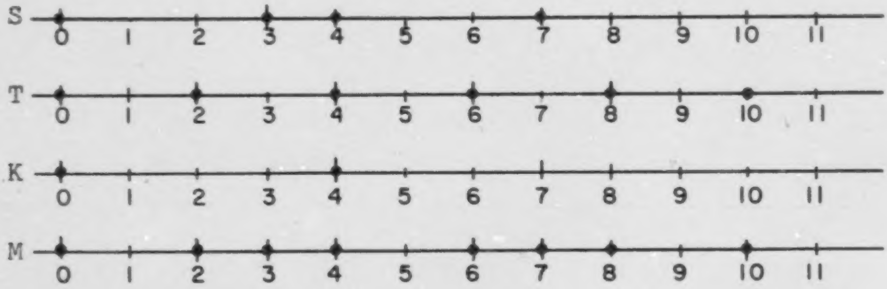
The number $\frac{22}{7}$ is between the whole numbers 3 and 4. $\frac{22}{7}$ is greater than 3.1. $\frac{22}{7}$ lies to the left of 3.2. $\frac{22}{7}$ lies between 3.1 and 3.2. Have the pupil divide 22 by 7; he should get the result 3.14285714...; from this he can decide that $\frac{22}{7}$ lies between 3.1 a and 3.2; between 3.14 and 3.15; between 3.142 and 3.143.

*7. Set S is one of three sets:
   (1) The set of whole numbers greater than 1.
   (2) The set of all counting numbers.
   (3) The set of all whole numbers.

Answers to Problem Set 1-2b; Pages 13-14:

1. (a) K = {0, 4}; M = {0, 2, 3, 4, 6, 7, 8, 10}

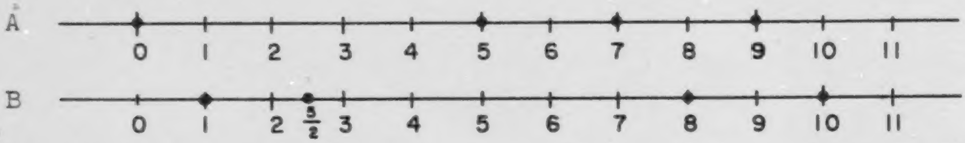
   (b)



   (c) Any point on the graphs of both S and T is on the graph of K. Any point on the graph of either S or T or both is a point on the graph of M.

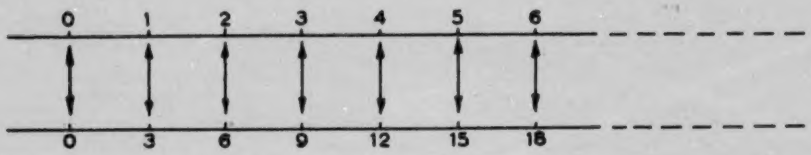
This exercise suggests that it will sometimes be convenient to compare the graphs of two sets to find the important sets later to be known as the intersection and the union of the sets.

2.



Since no point on both graphs has the same coordinate, the set C is the empty set, ∅.

*3.


0
1
2
3
4
5
6
0
3
6
9
12
15
18

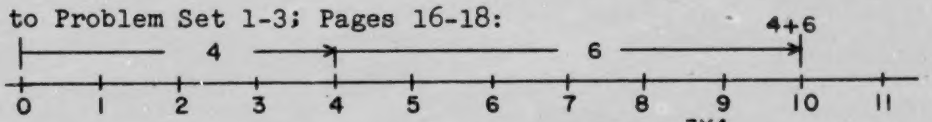
The student who does this problem can take from it a means of identifying an infinite set, for only an infinite set has the property that its elements may be placed in one-to one-correspondence with the elements of some proper subset of itself.

1-3. Addition and Multiplication on the Number Line

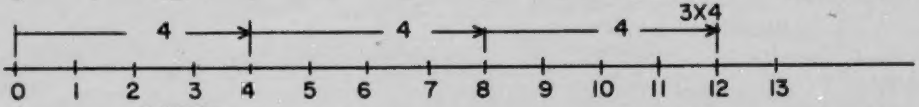
Page 15. The sum of two rational numbers is always a rational number. The closure property for addition of rational numbers is considered at greater length in Chapter 3; for the moment we rely upon the student's intuition, bolstered by some trial additions of rational numbers, to lead him in the right direction.

Answers to Problem Set 1-3; Pages 16-18:

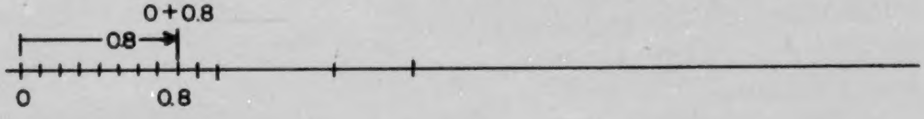
1. (a)


4+6
4
6
0
1
2
3
4
5
6
7
8
9
10
11

(b)

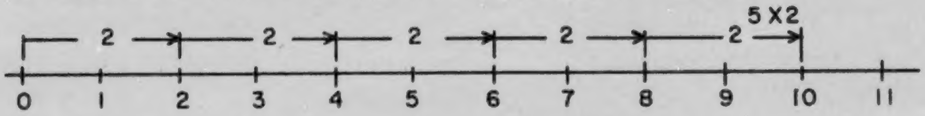

3X4
4
4
4
0
1
2
3
4
5
6
7
8
9
10
11
12
13

(c)

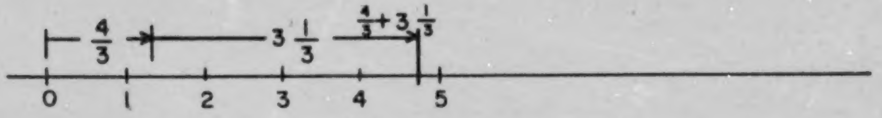


This problem may be interpreted as moving from 0 to 0 on the number line and then moving 0.8 units to the right.

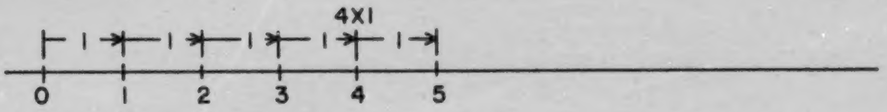
(d)


5X2
2
2
2
2
2
0
1
2
3
4
5
6
7
8
9
10
11

(e)



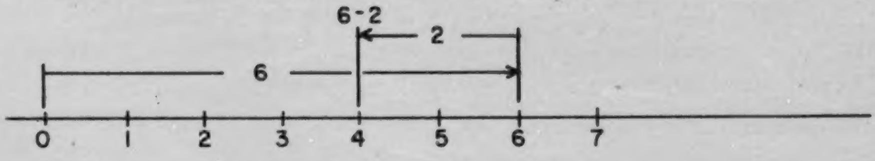
(f)



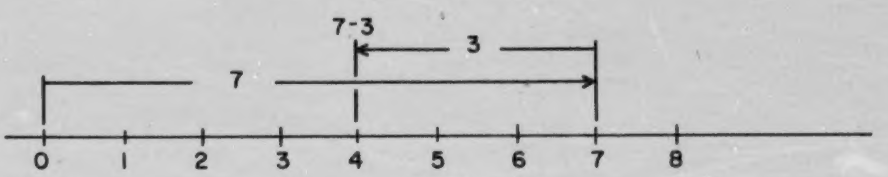
An acceptable alternate beginning for the procedure for addition on the number line is to locate the point whose coordinate is the first number directly, rather than moving from 0 to that number on the number line.

2. A procedure for subtracting two numbers on the number line is: move from 0 to the first number on the number line, and then move from this point to the left the number of units given by the second number, thus locating the point whose coordinate is the original number.

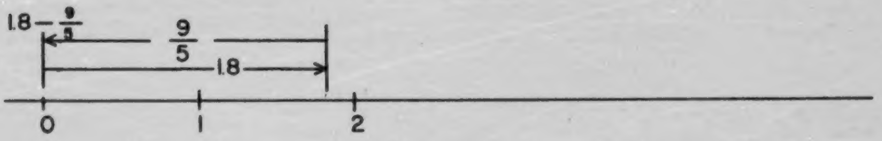
(a)



(b)

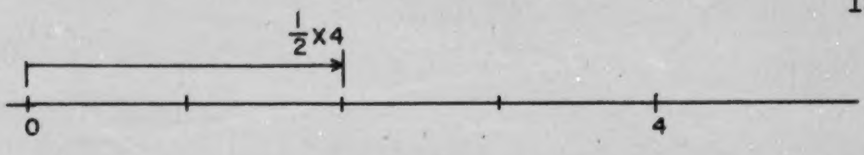


*(c)



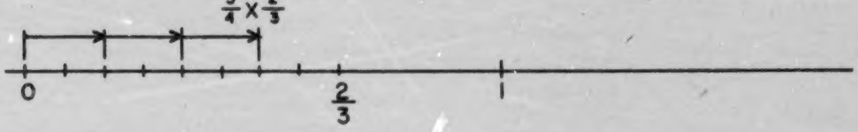
Using this procedure on other problems may involve some students in negative numbers. Discussion of this question should be kept to the minimum necessary to placate the student, for subtraction is dealt with thoroughly, for all real numbers, in Chapter 9. We don't propose merely to forget that the student knows how to subtract positive numbers until then, but we will casually use subtraction in many exercises in the earlier chapters without, however, discussing the operational properties of subtraction.

*3. (a)



The segment from 0 to 4 is divided into two equal parts, and at the right end of the first segment is the required point.

(b)



The segment from 0 to $\frac{2}{3}$ is divided into four equal parts, and at the right end of the third segment is the required point.

The student may use the fraction $\frac{8}{12}$ in place of $\frac{2}{3}$, and thus find simpler names for the coordinate of the point at the right of the third equal segment, such as $\frac{6}{12}$ or $\frac{1}{2}$.

4. (a) The sum of any two odd numbers is an even number.
   $5 + 13 = 18$
   (b) The product of any two odd numbers is an odd number.
   $5 \times 13 = 65$

This problem is included in the exercises to help keep informally before the student the idea of closure. The formal definition of closure is found in Problem *7 of these exercises and again in the text in Chapter 3.

5. $S = \{2, 3, 4, 5, 6, 7, 8\}$.
   S is not a subset of T.

If the student has difficulty finding the sum of all pairs of elements of the set in a problem of this type, the use of an addition table may help him systematize his work:

+	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

In such a table, a number in the vertical column on the left is the first number in each pair, while a number from the horizontal column on top is the second number of the pair.

The student should notice that no element is repeated in the listing of the set.

6. P = {0, 1}
   P is a subset of Q

   As in problem 5 above, a table may be helpful:

×	0	1
0	0	0
1	0	1

*7. **(a) Q is not closed under addition**
   (b) Yes.
   No, $\frac{2+4}{2} = 3$
   (c) Yes.
   Yes.
   (d) Yes.
   Yes.

*8. (a) The set of all even numbers is closed under the operation "twice the product". Other infinite sets of even numbers are also closed under this operation, e. g.,

{0, 2, 4, 8, 16, . . .}
{0, 4, 8, 16, ... }
{8, 16, 32, . . }

as well as the sets {0} and ∅.

(b) The set of all odd numbers is closed under the operation "twice the product and add one" also other infinite sets of odd numbers, e. g.

{3, 5, 7, 9, ...}
9, 11, 13, ...}

9. These seem to be the important ideas of the chapter. They may be listed in any order, though here they appear in order of their first appearance in the text.

[page 18]

Section 1-1.

1. A *set* is a collection of elements or members with some common characteristic.

2. The set which contains no elements is called the *empty set*, or *null set*, and is denoted by the symbol $\emptyset$.

3. If every element of set A belongs to set B, then A is a *subset* of B.

4. If the elements of a set can be counted, with the counting coming to an end, the set is a *finite* set. Otherwise, it is an *infinite* set. The only exception to this rule is the set $\emptyset$, which cannot be counted, but it is nevertheless finite.

Section 1-2.

5. Every whole number and every counting number has a *successor*, so that the set of all whole numbers and the set of all counting numbers are infinite sets.

6. A *fraction* is a symbol which indicates the quotient of two numbers.

7. A number which *can be* represented by a fraction indicating the quotient of two whole numbers, excluding division by zero, is a *rational number*.

8. There are many possible names for the same number.

9. Each rational number may be *associated* with a point on the number line.

10. The set of all whole numbers is a *subset* of the set of all rational numbers.

11. The *coordinate* of a point is the number associated with the point on the number line.

12. Every number may be associated with a point on the number line, and every point on the number line corresponds to a number--though not necessarily a rational number.

13. There are infinitely many points on the number line; there are infinitely many points between each pair of points on the number line.

[page 18]

14. The graph of a set is the set of points whose coordinates are elements of the set.

15. Picturing addition on the number line reveals that adding the second of two numbers to the first is a different process from adding the first to the second, although both processes terminate at the same point on the line.

16. Picturing multiplication on the number line reveals that multiplying the second of two numbers by the first is a different process from multiplying the second by the first, although both processes terminate at the same point on the line.

CHAPTER 1

Suggested Test Items

(The "suggested test items" which follow are not intended to comprise a balanced or complete test, but are, as the title implies, questions which seem suitable for inclusion in a test on this chapter.)

1. Are the following sets finite or infinite? If it is possible, list the members of each.
   (a) The number of people in this classroom today.
   (b) all multiples of 3.
   (c) all counting numbers less than 7.
   (d) all whole numbers which are not multiples of 5.
   (e) all numbers between 0 and $\frac{1}{4}$.
2. (a) Given set S = {0, 1, 2, 3, 4}. Find set T, the set of products of each element of set S and 1. Is T a subset of S?
   (b) Given set A = {0, 2, 4, 6, 8}. Find set B, the set of products of each element of set A and 0. Is B a subset of A?
3. List all the subsets of set Z = {1, 3, 4, 5}. How many of them have 3 elements?

[page 18]

4. Given set N = {1, 2, 3, 4, 8, 9, 12, 16}.
   - (a) Find the subset R consisting of all members of N which are squares of whole numbers.
   - (b) Find set K of the odd numbers in set N.
   - (c) Find set A of the squares of the members of N.
   - (d) Find set B whose members are each 3 more than twice the corresponding member of N.
   - (e) Find set C, the set of all numbers which are elements of both N and B.
   - (f) Find set D, the set of all numbers which are elements of either N or B or both.
5. Consider each of the following sets, and for those which are finite list the elements.
   - (a) All counting numbers less than 1.
   - (b) All whole numbers less than 1.
   - (c) All numbers less than 1.
   - (d) All counting numbers such that 10 times the number is greater than the number itself.
   - (e) All whole numbers such that 10 times the number is equal to the number times itself.
6. On the number line, locate the points whose coordinates are:
   - (a) $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \frac{9}{3}$.
   - (b) 0, .6, 1.2, 1.5, 1.8, 2, 3.5, 4.

   Which of these are counting numbers? Whole numbers? Rational numbers? On the number line, how is the point with coordinate 5.4 located with respect to the point with coordinate 4? with coordinate 6.2?
7. (a) Is $\frac{3}{7}$ to the left of $\frac{15}{21}$ on the number line?
   - (b) Show the graph of the set K = {0, 3, 7}.
   - (c) Write 3 other names that could be used for the coordinate 3.
8. If A is the set of all whole numbers less than 20 which are not multiples of either 2, 3, or 5.
   - (a) List the elements of set A .
   - (b) Draw the graph of set A .

9. List two numbers between $\frac{1}{8}$ and $\frac{2}{8}$. How do you know that they are between $\frac{1}{8}$ and $\frac{2}{8}$?

10. Perform the following operations on the number line:

    (a) $4 + 2$ (c) 1×5 (e) 3×0.5

    (b) 4×2 (d) $1\frac{1}{2} + \frac{5}{2}$

*11. State S, the set of all whole numbers.

    (a) Is it finite or infinite?
    (b) Is it closed under addition? Explain why.
    (c) Is it closed under multiplication? Explain why.
    (d) Is it closed under "average"? Show why.

    State T, the set of all odd numbers, and answer questions (a) through (d).

    State R, the set of all odd numbers less than 8 and answer questions (a) through (d).

Chapter 2

NUMERALS AND VARIABLES

For background in the topics included in this chapter, the teacher is referred to Haag, Studies in Mathematics, Volume III, Structure of Elementary Algebra, Chapter 3, Sections 1 and 2, and Chapter 6, Section 1.

Students who have studied the 7th Grade S.M.S.G. material will have been exposed to the properties of addition and multiplication for whole numbers and for rational numbers. If they have studied the 8th Grade S.M.S.G. material they will have seen the properties for real numbers.

Such students should probably find Sections 2-2 and 2-3 a review which they can do quickly. At the same time you should help them avoid the danger of being overconfident and missing the deeper significance of these concepts. Keep them on the watch for any ideas which they have not already seen in their earlier work.

While these S.M.S.G. students have been using variables in the 7th and 8th grades, they have been doing so rather casually. The careful treatment of variable in Section 2-4 should be studied well enough to try to establish a clear understanding of this concept.

2-1. Numerals and Numerical Phrases.

The aim of this section is to bring out the distinction between numbers themselves and the names for them and also to introduce the notion of a phrase. Along the way, a number of important conventions used in algebra are pointed out.

Page 19. We do not want to make a precise definition of "common name." The term is a relative one and should be used quite informally. Note that some numbers, such as $\pi\sqrt{2}$, do not have what we would wish at this time to call a common name, while some may have several common names (e.g. $\frac{1}{2}$, 0.5 and $2\frac{1}{2}$, $\frac{5}{2}$, etc.).

The ideas of indicated sum and indicated product are very handy, particularly in discussing the distributive property, and will be used frequently. They are also useful to counteract the tendency, encouraged in arithmetic, to regard an expression such as "4 + 2" not as the name of a number but rather as a command to add 2 to 4 to obtain the number 6. This point of view makes it difficult for a pupil to accept such expressions as names of anything. In passing, you may wish to mention to the class indicated quotients and indicated differences. Some may already be familiar with "indicated quotient" as synonymous with "fraction."

Note the use of quotes to indicate when the reference is to the numeral or expression rather than to the number represented. It is important to be careful about this at first. However, since good English does not always demand this kind of distinction but rather allows the context to give the meaning, we tend later to become more relaxed about it and use such forms as "the expression 3x - 4y + 7" rather than "the expression '3x - 4y + 7'."

Page 20. The agreement about the preference for multiplication over addition is made to facilitate the work with expressions and not as an end in itself. In certain kinds of expressions the agreement should also apply to division as well as multiplication, for example when division is written in the form 2/3 or 2 ÷ 3 rather than $\frac{2}{3}$. We prefer to avoid these forms and, in particular, to discourage the use of the symbol " ÷."

The use of parentheses might be compared to the use of punctuation marks in the writing of English. Emphasis should be on the use of parentheses to enable us to read expressions without ambiguity and not on the technique of manipulating parentheses for their own sake.

Answers to Problem Set 2-1a; page 21:

1. (a) Other numerals for 8 include $\frac{16}{2}$, $3\cdot\frac{8}{3}$, 2(2 +2), 5 + 3, $\frac{8}{1}$, $8\cdot1$, 4(7 - 5), $\frac{10(3 - 1)}{2.5}$, 3 + 5, $1\cdot8$.

   The different numerals given in answer to this problem will point to some of the properties of addition and multiplication which are coming. Do not, however, stress them at this point.

   (b) There are infinitely many numerals for 8.

2. (a) Yes (c) No (e) No (g) Yes

   (b) Yes (d) Yes (f) Yes (h) Yes

Note that exercises (f) and (h) point toward the associative property for addition, which is discussed in the next section.

3. (a) 17
(b) 24
(c) 133
(d) 20
(e) 34
(f) 39
(g) 13
(h) 23
(i) $\frac{20}{7}$
(j) $\frac{4}{3}$
(k) 2
(l) 3

The words "numeral" and "numerical phrase" denote almost the same thing. A phrase may be a more complicated expression which involves some operations; "numeral" includes all these and also the common names of numbers. We do not wish to make any fuss over this distinction, and are happy if the student learns to use the words in this way in the course of the year just by watching others use them. We introduce both because people do use both, and because a term for a numeral which suggests some indicated operations is sometimes handy.

In the term "numerical phrase," the word "numerical" is not very important, and is used not so much to distinguish it from an English phrase as from an open phrase (one involving one or more variables) which is coming.

The word "operations" is intended at this point to suggest the basic operations of arithmetic (multiplication, division, addition and subtraction.) In some contexts it may be desirable to admit operations such as the square root, forming the absolute value, etc.

Page 22. The words "true" and "false" for sentences seem preferable to "right" and "wrong" or "correct" and "incorrect" because the latter all imply moral judgments to many people. There is nothing illegal, immoral, or "wrong" in the usual sense of the word about a false sentence. The student should be encouraged to use only "true" and "false" in this context.

We have been doing two kinds of things with our sentences: We talk about sentences, and we use sentences. When we write

"3 + 5 = 8" is a true sentence,

we are talking about our language; when, in the course of a series of steps, we write

$$3 + 5 = 8,$$

we are _using_ the language. When we talk about the language, we can perfectly well talk about a false sentence, if we find this useful. Thus it is quite all right to say

"$3 + 5 = 10$" is a false sentence;

but it is far from all right to use the sentence

$$3 + 5 = 10$$

in the course of a proof. When we are actually _using_ the language, false sentences have no place; when we are talking about our language, they are often very useful.

Answers to Problem Set 2-1b; pages 22 - 23:

1. (a) False
   (b) True
   (c) True
   (d) False
   (e) False
   (f) True
   (g) True
   (h) False
   (i) True
   (j) False
   (k) False

   Note that parts (b) and (c) point to the distributive property, part (g) to the commutative property for multiplication, both of which are considered in the next section.

2. (a) 13
   (b) 58
   (c) $\frac{32}{5}$
   (d) 4
   (e) 16
   (f) 15

3. (a) $(\frac{1}{2} \times 6) + 3$
   (b) $(2\cdot5) + (6\cdot2)$; or $(2\cdot5) + 6\cdot2$; or $2\cdot5 + (6\cdot2)$
   (c) $(2 \times 3) + (4 \times 3)$; or $(2 \times 3) + 4 \times 3$; or $2 \times 3 + (4 \times 3)$
   (d) $(3 \times 8) - 4$

4. (a) $10 - (7 - 3) = 6$
   (b) $3\cdot(5 + 7) = 36$
   (c) $(3\cdot5) + 7 = 22$
   (d) $3\cdot(5 - 4) = 3$
   (e) $(3\cdot5) - 4 = 11$

(f) $(3 \times 5) + (2 \times 4) = 23$
(g) $3 \times (5 + 2) \times 4 = 84$
(h) $(3 \times 5 + 2) \times 4 = 68$; or $\big((3 \times 5) + 2\big) \times 4 = 68$
(i) $3 \times (5 - 2) \times 4 = 36$
(j) $(3 \times 5) - (2 \times 4) = 7$
(k) $(3 \times 5 - 2) \times 4 = 52$; or $\big((3 \times 5) - 2\big) \times 4 = 52$
(l) $(12 \times \frac{1}{2} - \frac{1}{3}) \times 9 = 51$; or $\big((12 \times \frac{1}{2}) - \frac{1}{3}\big) \times 9 = 51$
(m) $(12 \times \frac{1}{2}) - (\frac{1}{3} \times 9) = 3$
(n) $12 \times (\frac{1}{2} - \frac{1}{3}) \times 9 = 18$
(o) $(3 + 4)(6 + 1) = 49$
(p) $3 + 4(6 + 1) = 31$
(q) $(3 + 4)6 + 1 = 43$
(r) $(3 + 4 \cdot 6) + 1 = 28$; or $3 + (4 \cdot 6 + 1) = 28$

2-2. Some Properties of Addition and Multiplication.

The aim of this and the next section is to look at the fundamental properties of addition and multiplication in terms of specific numbers. We go as far as obtaining a general statement of the properties in English. You should not state the properties at this time using variables. We do not need these formulations at this point and prefer to lead up to variables in a different way in Section 2-4. It is important to emphasize the pattern idea here and you may want to do this by writing something like the following on the board when discussing, for example, the associative property for addition:

(first number + second number) + third number =
first number + (second number + third number).

The use of the properties of addition and multiplication as an aid to computation in certain kinds of arithmetic problems is both interesting and important, but is not the main point of these properties. These properties will play a much more fundamental role in this course. They constitute the foundation on which the entire subject of algebra is built.

We shall return to the properties in Chapter 3 where the general statements using variables will be given. They are discussed here not only as a part of the "spiral method" but because the distributive property is needed in introducing the concept of variable.

Page 24. $(1.2 + 1.8) + 2.6$ leads to the simpler second addition:

$$3.0 + 2.6$$

When the operations $5 + 3$ and $3 + 5$ were performed on the number line, both processes terminated at the same point of the number line. $5 + 3$ and $3 + 5$, then, are names for the same coordinate, and numerals for the same number. In Chapter 1, you may have preferred to think of $5 + 3$ as starting from 5 and moving three to the right. This point of view is still correct.

Answers to Problem Set 2-2a; page 26:

One purpose of this problem set is to help the students make a habit of quickly recognizing addition combinations which facilitate computation. Another and more immediate purpose is to help the pupils begin to become aware that these manipulations, which they may have taken for granted, are possible because of the associative and commutative properties of addition.

In these problems we do not ask specifically which properties are used in going from one step to the next. This is often tedious, particularly in the latter steps of the calculation. We do not insist that this be done now, for we are more concerned at this time with having the student recognize the usefulness of the properties than in having him pursue a thorough step-by-step reasoning process from beginning to end of the calculation.

In several parts of this problem there are variations on "the easiest way" to perform the additions. Comparison of some of these in class discussion should help fulfill the purpose of the question.

1. (a) The student may express his answer in a manner like this: "Add the 6 and the 4 to get 10, then 10 and 8 to make 18." This can be shown step by step in several ways, e.g.:

$6 + (8 + 4)$
$= 6 + (4 + 8)$
$= (6 + 4) + 8$
$= 10 + 8$
$= 18$

$6 + (8 + 4)$
$= (8 + 4) + 6$
$= 8 + (4 + 6)$
$= 8 + 10$
$= 18$

(b) $\frac{2}{5} + \frac{2}{3} + 1 + \frac{1}{3} + \frac{8}{5}$
$= \frac{2}{5} + \frac{8}{5} + \frac{2}{3} + \frac{1}{3} + 1$
$= (\frac{2}{5} + \frac{8}{5}) + (\frac{2}{3} + \frac{1}{3}) + 1$
$= 2 + 1 + 1$
$= 2 + (1 + 1)$
$= 2 + 2$
$= 4$

(c) $5\frac{4}{7} + 6 + 14\frac{3}{7}$
$= 5\frac{4}{7} + 14\frac{3}{7} + 6$
$= (5\frac{4}{7} + 14\frac{3}{7}) + 6$
$= 20 + 6$
$= 26$

(d) $\frac{3}{4} + \frac{1}{3}$ This is a case in which there is no "easiest" way. Neither property is of help in this computation, though several properties to be studied later, most notably the distributive property, lie behind the student's calculations.

$\frac{3}{4} + \frac{1}{3} = \frac{13}{12}$

(e) $2\frac{1}{5} + 3\frac{2}{3} + 6 + 7\frac{4}{5}$
$= 2\frac{1}{5} + 7\frac{4}{5} + 6 + 3\frac{2}{3}$
$= (2\frac{1}{5} + 7\frac{4}{5}) + 6 + 3\frac{2}{3}$
$= 10 + 6 + 3\frac{2}{3}$
$= (10 + 6) + 3\frac{2}{3}$
$= 16 + 3\frac{2}{3}$
$= 19\frac{2}{3}$ or $\frac{59}{3}$

[page 26]

(f) Here is another case in which neither property facilitates the computation. $\frac{10}{3} + \frac{6}{5} = \frac{68}{15}$

(g) $(1.8 + 2.1) + (1.6 + .9) + 1.2$
$= 1.2 + (1.8 + 2.1) + (.9 + 1.6)$
$= (1.2 + 1.8) + (2.1 + .9) + 1.6$
$= 3.0 + 3.0 + 1.6$
$= (3.0 + 3.0) + 1.6$
$= 6.0 + 1.6$
$= 7.6$

(h) $(8 + 7) + 4 + (3 + 6)$
$= (8 + 7) + (3 + 6) + 4$
$= 8 + (7 + 3) + (6 + 4)$
$= 8 + 10 + 10$
$= 8 + (10 + 10)$
$= 8 + 20$
$= 28$

2. 179 millimeters
Yes

Although the question is very easy to answer, the fact that the answer is "yes" depends upon the two properties studied in this section which tell the student that numbers may be added in any order. Thus

$$32 + 71 + 76 = 76 + 71 + 32.$$

Page 27. The phrase $(4 \times 1.5) \times 3$ is easier to handle because the first multiplication yields a whole number, simplifying the next step.

The word "adjacent" in line 22 is one we would like the students to learn.

Another interchange would be to commute 7×4 to 4×7 in the phrase on the right, which would then read $\frac{3}{4}(4 + 7)$. Now the example is in the form

$$(7 \times \tfrac{3}{4}) \times 4 = \tfrac{3}{4} \times (4 \times 7)$$

Applying the associative property, we have

$$7 \times (\tfrac{3}{4} \times 4) = (\tfrac{3}{4} \times 4) \times 7$$
$$7 \times 3 = 3 \times 7$$

Applying the commutative property again, we have

$$7 \times 3 = 7 \times 3 \;.$$

Answers to Problem Set 2-2b; page 28:

1. This problem is intended to serve the same purposes for the properties of multiplication which Problem 1 of Problem Set 2-2a served for the addition properties.

(a)
$$\begin{aligned} & 4 \times 7 \times 25 \\ &= 4 \times 25 \times 7 \\ &= (4 \times 25) \times 7 \\ &= 100 \times 7 \\ &= 700 \end{aligned}$$

(b)
$$\begin{aligned} & \tfrac{1}{5} \times (26 \times 5) \\ &= \tfrac{1}{5} \times (5 \times 26) \\ &= (\tfrac{1}{5} \times 5) \times 26 \\ &= 1 \times 26 \\ &= 26 \end{aligned}$$

$$\begin{aligned} & \tfrac{1}{5} \times (26 \times 5) \\ &= (26 \times 5) \times \tfrac{1}{5} \\ &= 26 \times (5 \times \tfrac{1}{5}) \\ &= 26 \times 1 \\ &= 26 \end{aligned}$$

(c)
$$\begin{aligned} & 73 + 62 + 27 \\ &= 73 + 27 + 62 \\ &= (73 + 27) + 62 \\ &= 100 + 62 \\ &= 162 \end{aligned}$$

This problem is a reminder that addition properties are not to be forgotten while multiplication properties are at the center of attention.

(d)
$$\begin{aligned} & (3 \times 4) \times (7 \times 25) \\ &= 3 \times (4 \times (7 \times 25)) \\ &= 3 \times ((7 \times 25) \times 4) \\ &= 3 \times (7 \times (25 \times 4)) \\ &= 3 \times (7 \times 100) \\ &= (3 \times 7) \times 100 \\ &= 21 \times 100 \\ &= 2100 \end{aligned}$$

The student will probably give an answer such as "Multiply 25 by 4 and get 100; then multiply 3 by 7 and get 21; then multiply 100 by 21 and get 2100."

(e) Here is an exercise in which there is no "easiest way," that is, regrouping is not involved.

$12 \times 14 = 168$

(f) $\frac{1}{2} \times \frac{1}{3} \times \frac{5}{6}$

$= (\frac{1}{2} \times \frac{1}{3}) \times \frac{5}{6}$

$= \frac{1}{6} \times \frac{5}{6}$

$= \frac{5}{36}$

This way of doing the calculation is preferable only in that it involves only one digit numbers until the simplest form is written.

(g) $6 \times 8 \times 125$

$= 6 \times (8 \times 125)$

$= 6 \times 1000$

$= 6000$

(h) $(1.25) \times 5.5 \times 8$

$= 5.5 \times 8 \times 1.25$

$= 5.5 \times (8 \times 1.25)$

$= 5.5 \times 10$

$= 55$

(i) $(2 \times 5) \times 1.97$

$= 10 \times 1.97$

$= 19.7$

Observe that in this case the original form of the problem is the best from which to work.

(j) $\frac{5}{4} \times 6 \times \frac{4}{3} \times \frac{1}{5}$

$= 6 \times \frac{4}{3} \times \frac{1}{5} \times \frac{5}{4}$

$= 6 \times \frac{4}{3} \times (\frac{1}{5} \times \frac{5}{4})$

$= 6 \times \frac{4}{3} \times \frac{1}{4}$

$= 6 \times (\frac{4}{3} \times \frac{1}{4})$

$= 6 \times \frac{1}{3}$

$= 2$

2. The first forms of each part of the problem are easier to compute because repetition of a partial product is involved in each case. Thus the recurring partial products can be copied after their first writing.

Page 31. The aim of these exercises, in the cases in which the sentences are true, is to have the student recognize their truth, not because both sides can be reduced to the same common name, but because the sentence is an example of a true pattern. You may have to remind your students to do this. For instance, in Problem 1(b), the best answer the student could give would be: "I know that $15 = 7 + 8$, and the sentence follows the pattern of the distributive property. Hence it's true." A form of work as shown below also gets at the same point and is perfectly acceptable. The false sentences are intended to catch some mistakes students might make through misstatement of the distributive property, (d), or misuse of the notation for mixed fractions, (f).

Answers to Problem Set 2-3a; page 31:

1. (a) $2 \times 4 + 5 \times 4 = 7 \times 4$
$= (2 + 5)4$
$= (7)(4)$ True

(b) $15 \times 2 = 7 \times 2 + 8 \times 2$
$= (7 + 8)2$
$= (15)2$ True

(c) $25(40 + 3) = 25(40) + 25(3)$
$25(40) + 25(3) = 25(40) + 25(3)$ True

(d) $3(2) + 6(3) = 9(6)$
$3(2) + 3(6) = 9(6)$
$3(2 + 6) = 9(6)$
$3(8) = 9(6)$
$24 = 54$ False

(e) $13(19 + 1) = 13(19) + 13(1)$
$13(19) + 13(1) = 13(19) + 13(1)$ True

(f) $2(\frac{1}{2} + \frac{1}{4}) = 2\frac{1}{2} + 2\frac{1}{4}$
$2(\frac{1}{2}) + 2(\frac{1}{4}) = 2\frac{1}{2} + 2\frac{1}{4}$ $1\frac{1}{2} = 2\frac{1}{2} + 2\frac{1}{4}$
$1 + \frac{1}{2} = 2\frac{1}{2} + 2\frac{1}{4}$ False

This problem should recall the discussion of indicated multiplication, and remind the student that the "mixed number" $2\frac{1}{2}$ does not represent $2(\frac{1}{2})$.

(g) $12(\frac{2}{3} + \frac{3}{4}) = 12(\frac{2}{3}) + 12(\frac{3}{4})$

$12(\frac{2}{3}) + 12(\frac{3}{4}) = 12(\frac{2}{3}) + 12(\frac{3}{4})$ True

(h) $3(2.5) + 3(1.5) = 3(2.5 + 1.5)$

$3(2.5 + 1.5) = 3(2.5 + 1.5)$ True

The extra words distributive property for multiplication over addition are not used very often in the rest of the course, but they may be handy for reference when students perform incorrect manipulations. It is not necessary that the student grasp the significance immediately.

In the third example the indicated product is the form which facilitates computation. This form is also easier in Problem 1 (a), (b), (d), (e), (h) on page 13. The indicated sum is easier in Problem 1 (c), (f), and (g).

Page 32. $(\frac{1}{3} + \frac{1}{4})12 = 12(\frac{1}{3} + \frac{1}{4})$ by the commutative property of multiplication.

The detailed discussion of

$$(\frac{1}{3} + \frac{1}{4})12 = (\frac{1}{3})12 + (\frac{1}{4})12$$

is really, among other things, a preparation for the proofs to come later in the course. We are not content just to check the truth of this sentence by arithmetic operations; we find out that this sentence is true also by an argument from a short series of sentences which we have previously seen to be true. There is no need to dwell upon this development at the present time. It is the first of many to come.

Page 33. A pattern suggested is: Given three numbers, the product of the first and the second plus the product of the third and the second is the same number as the product obtained when the sum of the first and third is multiplied by the second.

Answers to Problem Set 2-3b; page 33:

1. (a) $12(3 + 4) = 12(3) + 12(4)$

   (b) $3(5) + 3(7) = 3(5 + 7)$

   (c) $(2.5 + 4.5)4 = (2.5)4 + (4.5)4$

   (d) $24(\frac{1}{8}) + 24(\frac{1}{6}) = 24(\frac{1}{8} + \frac{1}{6})$

   (e) $7(17) + 6(17) = 13(17)$

   (f) $(3 + 11)2 = 3(2) + 11(2)$

   This problem is a true sentence no matter which number we may use. This prepares for the precise statement of the distributive property in the next chapter. Do not use a variable at this time in discussing this with the students.

2. (a) $27(\frac{7}{8}) + 27(\frac{1}{8}) = 27(\frac{7}{8} + \frac{1}{8})$
   $= 27(1)$
   $= 27$

   (b) $\frac{1}{3}(12) + \frac{1}{4}(12) = (\frac{1}{3} + \frac{1}{4})12$
   $= 4 + 3$
   $= 7$

   (c) $(.36 + .14).6 = .36(.6) + .14(.6)$
   $= (.50).6$
   $= .30$

   (d) $12(5 + 5) = 12(5) + 12(5)$
   $= 12(10)$
   $= 120$

   (e) $(2.3 + 4.6) + 7.7 = (2.3 + 7.7) + 4.6$
   $= 10 + 4.6$
   $= 14.6$

   (f) $6(\frac{3}{2} + \frac{2}{3}) = 6(\frac{3}{2}) + 6(\frac{2}{3})$
   $= 9 + 4$
   $= 13$

   (g) $.71(.8) + .2(.71) = .71(.8) + .71(.2)$
   $= .71(.8 + .2)$
   $= .71(1.0)$
   $= .71$

(h) $3(2 + 7 + 6 + 5) = 3(2) + 3(7) + 3(6) + 3(5)$
$= 3(20)$
$= 60$

3. The area of triangle ACD is equal to the area of triangle EFI plus the area of triangle FGH. We have learned that the number of square units in the area of a triangle is one-half the product of the number of units in the base and the number of units in the altitude. The three triangles used here all have the same altitude, 2.

Area of triangle ACD = $\frac{1}{2}(4)(2)$

Area of triangle EFI = $\frac{1}{2}(3)(2)$

Area of triangle FGH = $\frac{1}{2}(1)(2)$

$\frac{1}{2}(4)(2) = \frac{1}{2}(3)(2) + \frac{1}{2}(1)(2)$

becomes

$\frac{1}{2}(2)(4) = \frac{1}{2}(2)(3) + \frac{1}{2}(2)(1)$ (commutative property for multiplication)

$\frac{1}{2}(2)(4) = \frac{1}{2}(2)(3 + 1)$ (distributive property)

From this statement it is apparent that when the altitudes of two triangles are the same, their respective areas are dependent on their bases.

4. $(\frac{1}{2} + \frac{2}{3})11 + (\frac{1}{2} + \frac{2}{3})7 = (\frac{1}{2} + \frac{2}{3})(11 + 7)$

$= (\frac{1}{2} + \frac{2}{3})18$

$= \frac{1}{2}(18) + \frac{2}{3}(18)$

$= 9 + 12$

$= 21$

By using the distributive property twice the student can avoid inconvenient addition. This problem is important preparation for the work with the distributive property in the next chapter, and should not be omitted.

5. (a)
$$\begin{aligned}
8(\tfrac{3}{5} + \tfrac{2}{3}) + (\tfrac{2}{3} + \tfrac{3}{5})7 &= 8(\tfrac{3}{5} + \tfrac{2}{3}) + 7(\tfrac{3}{5} + \tfrac{2}{3}) \\
&= (8 + 7)(\tfrac{3}{5} + \tfrac{2}{3}) \\
&= 15(\tfrac{3}{5} + \tfrac{2}{3}) \\
&= 15(\tfrac{3}{5}) + 15(\tfrac{2}{3}) \\
&= 9 + 10 \\
&= 19
\end{aligned}$$

(b)
$$\begin{aligned}
7(\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{3}{4}) + 5(\tfrac{5}{6} + \tfrac{3}{4}) &= 7\Big((\tfrac{1}{2} + \tfrac{1}{3}) + \tfrac{3}{4}\Big) + 5(\tfrac{5}{6} + \tfrac{3}{4}) \\
&= 7(\tfrac{5}{6} + \tfrac{3}{4}) + 5(\tfrac{5}{6} + \tfrac{3}{4}) \\
&= (7 + 5)(\tfrac{5}{6} + \tfrac{3}{4}) \\
&= 12(\tfrac{5}{6} + \tfrac{3}{4}) \\
&= 12(\tfrac{5}{6}) + 12(\tfrac{3}{4}) \\
&= 10 + 9 \\
&= 19
\end{aligned}$$

(*c)
$$\begin{aligned}
5(\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5}) + 7(\tfrac{1}{2} + \tfrac{1}{3}) &= 5\Big((\tfrac{1}{2} + \tfrac{1}{3}) + \tfrac{1}{5}\Big) + 7(\tfrac{1}{2} + \tfrac{1}{3}) \\
&= 5(\tfrac{1}{2} + \tfrac{1}{3}) + 5(\tfrac{1}{5}) + 7(\tfrac{1}{2} + \tfrac{1}{3}) \\
&= 5(\tfrac{1}{2} + \tfrac{1}{3}) + 7(\tfrac{1}{2} + \tfrac{1}{3}) + 5(\tfrac{1}{5}) \\
&= (5 + 7)(\tfrac{1}{2} + \tfrac{1}{3}) + 5(\tfrac{1}{5}) \\
&= 12(\tfrac{1}{2} + \tfrac{1}{3}) + 1 \\
&= 12(\tfrac{1}{2}) + 12(\tfrac{1}{3}) + 1 \\
&= 6 + 4 + 1 \\
&= (6 + 4) + 1 \\
&= 11
\end{aligned}$$

6. (a) $$\begin{aligned} 3 \times (17 + 4) \times \tfrac{1}{3} &= (3 \times \tfrac{1}{3})(17 + 4) \\ &= 21 \end{aligned}$$

(b) $$\begin{aligned} \frac{3(17 + 4)}{3} &= 17 + 4 \\ &= 21 \end{aligned}$$

(c) $$\begin{aligned} \tfrac{1}{5}(3 + 8)10 &= (\tfrac{1}{5})(10)(3 + 8) \\ &= 2(3 + 8) \\ &= 22 \end{aligned}$$

(d) $$\begin{aligned} 3(7) + 3(11) &= 3(7 + 11) \\ &= 3(18) \\ &= 54 \end{aligned}$$

(e) $$\begin{aligned} 3(7) + 3(2) &= 3(7 + 2) \\ &= 27 \end{aligned}$$

(f) $$\begin{aligned} 3(7) + 6 &= 21 + 6 \\ &= 27 \end{aligned}$$

(g) $$\begin{aligned} 7(4) + 42 &= 7(4) + 7(6) \\ &= 7(4 + 6) \\ &= 70 \end{aligned}$$

(h) $$\begin{aligned} 3(\tfrac{1}{5}) + \tfrac{2}{5} &= 3(\tfrac{1}{5}) + 2(\tfrac{1}{5}) \\ &= 5(\tfrac{1}{5}) \\ &= 1 \end{aligned}$$

(i) $$\begin{aligned} 3(17) + 12 &= 3(17) + 3(4) \\ &= 3(17 + 4) \\ &= 63 \end{aligned}$$

(j) $$\begin{aligned} 6(19) + 19 &= (6 + 1)19 \\ &= 133 \end{aligned}$$

(k) $$\begin{aligned} (10 + 2) \times 4 \times \tfrac{1}{2} &= (10 + 2) \times (4 \times \tfrac{1}{2}) \\ &= 12 \times 2 \\ &= 24 \end{aligned}$$

(l) $$\begin{aligned} \frac{(10 + 2) \times 4}{2} &= \frac{12 \times 4}{2} \\ &= \frac{48}{2} \\ &= 24 \end{aligned}$$

Problems 6(a), 6(b), and 6(g) are very specific preparation for the discussion in Section 2-4, and should not be omitted.

Page 34. The aim of this section is to acquaint the pupil with one meaning of the word *variable*. At this point, we insist that "n" or "x", or whatever letter is used as the variable, must be *thought of* as the name of a definite number although we may not have very much information about that number. In some cases, such as in the exercise discussed in the text, the number may be unspecified because what we want to say about it is the same for every number in a given set. This is always the case when we are interested in the pattern or form of a problem rather than in the answer. In other cases, the number may be unspecified because we do not know what it is at the outset but will find it out later. Variables used in this context are usually called "unknowns." In any case, try to *avoid* the concept of a variable as something that varies over a set of numbers.

The set of numbers which includes all values of a variable is called "domain" by some, "range" by others, depending largely upon the point of view from which the variable and its set are being seen. There are points of view then, to support the choice of either term. Since the most natural connection for many teachers to make when a variable and its set is mentioned is to see that variable as the "independent" variable in a function relationship, the name "domain" for its set comes easily to mind. It must be emphasized, however, that the variable need not be seen as the "independent variable" in a function relationship, but may in fact be considered as the "dependent" variable.

The steps in the proposed exercise are

$$\begin{gathered} 7 \\ 3\cdot 7 \\ 3\cdot 7 + 3 \\ 2(3\cdot 7 + 3) \\ \frac{2(3\cdot 7 + 3)}{12} \end{gathered}$$

The last phrase is a numeral for 4.

Page 36. The student may wonder why we insist on writing $3(17) + 12 = 3(17 + 4)$ instead of $3(17) + 12 = 51 + 12 = 63$. Either method will, of course, lead to the same result. The first, however, prepares the way for the computation below using "n" instead of "17." Perhaps it will satisfy most students who raise the question if you point out that the first method brings out the pattern while the second method tends to obliterate the pattern.

The discussion of the exercise would not have been changed in any essential way if we had decided to denote the chosen number from S by some letter other than n.

Some teachers have found it helpful, in introducing the notion of variable to their students, to play a number game in class in addition to using the material in the text. It is possible to "stump" them orally with some "pick-a-number" games which they will not be able to penetrate by purely arithmetic means, after which they will readily appreciate the use of a variable. Another successful method has been to use such a game at the board.

Example: "Choose a number from some set S - like, for instance, the set of whole numbers between 1 and 30 - add 3, multiply by 2, and subtract twice the number chosen."

Different pupils try the game at the board with different numbers, and always obtain 6. Others may be instructed to leave the numerals in indicated form, another may use "number" instead of a specific numeral, and yet others may use a variable like "n" or "x" from the beginning. The board may look like this:

4	5	number	n
7	$5 + 3$	number + 3	$n + 3$
14	$2(5 + 3)$	2(number + 3)	$2(n + 3)$
or 14	$2 \cdot 5 + 6$	2(number) + 6	$2n + 6$
6	$2 \cdot 5 + 6 - 2 \cdot 5$	2(number) + 6 - 2(number)	$2n + 6 - 2n$
or 6	6	6	6

This example uses the distributive property, which the students have seen, but it also uses associativity with one subtraction, which they have not. The operations with numbers are quite simple, however, and so the "$2n - 2n$" may not give any trouble. It is certainly not worth making a fuss over. If the subtraction is, for some reason, likely to give trouble, the game may always be played with an example such as the one in the text which involves no subtraction.

Answers to Problem Set 2-4; page 38:

1. $2(t + 3)$
2. $\frac{2n + 5}{3}$
3. Both forms are correct. The second is found from the first by use of the associative property for multiplication.
4. $4y$
5. Neither form is correct. $2(a + b)$ and $2a + 2b$ are correct forms.
6. (a) $\frac{39}{2}$ (f) 0
   (b) 15 (g) 13
   (c) 27 (h) 0
   (d) $\frac{9}{4}$ (i) 9
   (e) $\frac{3}{2}$ (j) 10
7. (a)
$$\begin{aligned} \tfrac{9}{5}c + 32 &= \tfrac{9}{5}(85) + 32 \\ &= 153 + 32 \\ &= 185 \end{aligned}$$
   (b)
$$\begin{aligned} \frac{h(a + b)}{2} &= \frac{4(3 + 5)}{2} \\ &= 2(8) \\ &= 16 \end{aligned}$$
   (c)
$$\begin{aligned} p(1 + rt) &= 500\big(1 + 0.04(3)\big) \\ &= 500(1 + .12) \\ &= 500 + 60 \\ &= 560 \end{aligned}$$
   (d)
$$\frac{r\ell - a}{r - 1} \quad \frac{2(48) - 4}{2 - 1} = 96 - 4$$
$$= 92$$
   (e)
$$\begin{aligned} \ell wh &= 18(10)(6) \\ &= 1080 \end{aligned}$$

8. (a)
$$\begin{aligned}\frac{(3a + 4b) - 2c}{3} &= \frac{(3\cdot 3 + 4\cdot 2) - 2\cdot 4}{3}\\ &= 3\end{aligned}$$

(b)
$$\begin{aligned}\frac{(6a - 4b) + 5c}{5} &= \frac{(6\cdot 3 - 4\cdot 2) + 5\cdot 4}{5}\\ &= \frac{10 + 20}{5}\\ &= 6\end{aligned}$$

(c)
$$\begin{aligned}\frac{\left(\frac{7a}{2} + \frac{3b}{2}\right) - \frac{5c}{2}}{2} &= \frac{\left(\frac{7\cdot 3}{2} + \frac{3\cdot 2}{2}\right) - \frac{5\cdot 4}{2}}{2}\\ &= \frac{\frac{1}{2}(21 + 6) - \frac{20}{2}}{2}\\ &= \frac{\frac{7}{2}}{2}\\ &= \frac{7}{4}\end{aligned}$$

(d)
$$\begin{aligned}\frac{\left(1.5(3) + 3.7(2)\right) - 2.1(4)}{7} &= \frac{(4.5 + 7.4) - 8.4}{7}\\ &= \frac{11.9 - 8.4}{7}\\ &= \frac{3.5}{7}\\ &= .5\end{aligned}$$

9. Among the important ideas of this chapter are the following, which are written in order of first appearance in the chapter:

Section 2-1

1. Numerals are names for numbers, as distinguished from numbers themselves.
2. The presence of an equal sign between two numerals indicates that the numerals represent the same number.
3. In order that the meaning of the numeral shall not be in doubt, we agree to give multiplication preference over addition and subtraction, unless otherwise indicated, in expressions involving these operations.

4. Parentheses may be used to enclose those parts of an expression to be taken as a numeral.
5. A numerical phrase is any numeral given by an expression which involves other numerals along with signs for operations.
6. Numerical phrases may be combined to form numerical sentences, which may be either true or false, but not both.

Section 2-2

7. A property of an operation is "something which it has" or one of its characteristics.
8. A binary operation is one which is always performed upon two numbers. Addition is an example of a binary operation.
9. The associative property of addition.
10. The commutative property of addition.
11. The associative property of multiplication.
12. The commutative property of multiplication.

Section 2-3

13. The distributive property (of multiplication over addition.)
14. The distributive property has an alternate pattern, illustrated thus: $2(3 + 4) = 2(3) + 2(4)$ or $(3 + 4)2 = (3)2 + (4)2$.

Section 2-4

15. A variable is a numeral which represents a definite though unspecified number from a given set of admissible numbers.
16. A number which a given variable can represent is called a value of the variable.
17. The set of values of a variable is called its domain.

Chapter 2

Suggested Test Items

1. Insert parentheses in each of the following expressions so that the resulting sentence is true:
   - (a) $5 \times 4 + 3 = 35$
   - (b) $5 \times 4 + 3 = 23$
   - (c) $5 \times 4 + 3 \times 6 = 210$
   - (d) $5 \times 4 + 3 \times 6 = 110$
   - (e) $5 \times 4 + 3 \times 6 = 38$
   - (f) $5 \times 4 + 3 \times 6 = 138$
2. Tell which of the following sentences are true:
   - (a) $\frac{1}{2} + \frac{5}{8} = 1 + \frac{1}{8}$
   - (b) $\frac{1}{2} + \frac{1}{3} = \frac{1}{5}$
   - (c) $8(\frac{1}{2} + \frac{1}{4}) = 8(\frac{1}{2}) + \frac{1}{4}$
   - (d) $8(\frac{1}{2} + \frac{1}{4}) = 4 + 2$
3. Copy the following phrases and show the steps in finding the simplest name for the number described:
   - (a) $\frac{3(5 + 2) - 6}{5}$
   - (b) $\frac{8(4 + 3)}{4} - 3$
4. Which of the numerals listed below are names for 6:
   - (a) $\frac{6 + 12}{6}$
   - (b) $3(1 + 1)$
   - (c) $\frac{1 + (1 + 1) + 15}{6}$
   - (d) $\frac{1 + (3 - 3)}{\frac{1}{6}}$
5. Write three other names for the number 7.
6. Show how you would calculate the value of each phrase:
   - (a) $\frac{3}{4} + (\frac{1}{3})(6) + \frac{2}{3}$
   - (b) $1 + (2 + \frac{1}{3})3 + 1$
7. Using members of the set $\{2, 5, 7\}$, illustrate each of the following:
   - (a) the commutative property of addition
   - (b) the associative property of multiplication
   - (c) the distributive property

8. State the property illustrated by each of the following true sentences:

(a)	$3 + 8 + 7 = 3 + 7 + 8$	(d)	$35 + 42 = 7(5 + 6)$
(b)	$3(8 + 7) = 24 + 21$	(e)	$(3 + 2) + 5 = 3 + (2 + 5)$
(c)	$(5)(8)(9) = (8)(5)(9)$	(f)	$\frac{2}{3}\left((\frac{9}{4})(7)\right) = \left((\frac{2}{3})(\frac{9}{4})\right)(7)$

9. Show how you could use the associative, commutative, and distributive properties to perform each of the following computations as simply as possible:

(a) $6\frac{2}{5} + (17 + 3\frac{3}{5}) + 53$

(b) $\dfrac{5 \times 13 + 6 \times 13}{2 \times 11 + 9 \times 11}$

(c) $34.3(356) + 34.3(644)$

(d) $(3 \times 4)(7 \times 25)$

(e) $\frac{1}{5}(\frac{7}{8}) + \frac{1}{5}(\frac{1}{8})$

(f) $16(\frac{3}{8} + \frac{1}{4})$

(g) $42(5) + 40$

10. To add 24 + 66, one way is to say

$$\begin{aligned} 24 + 66 &= 24 + (6 + 60) \\ &= (24 + 6) + 60 \\ &= 30 + 60 \\ &= 90 \end{aligned}$$

(a) What property has been used?

(b) Add 24 + 66 in the way indicated below, and state the properties used:

$$\begin{aligned} 24 + 66 &= (20 + 4) + (60 + 6) \\ &= (\text{etc.}) \end{aligned}$$

11. Show how you use the associative and commutative properties to write most easily the common names for:

(a) $(5 \times 37) \times 20$

(b) $54 + 17 + 46$

(c) $3\frac{2}{3} + 4\frac{1}{7} + 6\frac{1}{3}$

12. Show the steps in finding the common name for $\frac{9(x + 3y) - z}{3}$ when x is 2, y is 0, and z is 1.

13. Find the value of $\frac{3(k + m) - p}{2m}$ if k is 5, m is 2, and p is 7.

14. A certain number n is multiplied by 5, then increased by 5, and the result is doubled. Which of the following best describes this statement?

    (a) $2 \times 5(n + 5)$　　(c) $2n(5 + 5)$

    (b) $2(5n + 5)$　　(d) $2(5n + 25)$

15. Write verbal directions which would result in the phrase $\frac{3n(n + 3)}{8}$.

16. A problem begins: "Pick a number from 1 to 10 inclusive ..." If n is the variable in this problem,

    (a) make a graph of the domain of n;

    (b) tell which of the following numbers could never be assigned to n: 2, $\frac{9}{4}$, $\frac{21}{2}$, π, $\frac{1}{3}$, 4π, 1.

Chapter 3

SENTENCES AND PROPERTIES OF OPERATIONS

This course is concerned with a systematic study of numbers and their properties. Arithmetic would seem at first to have had the same purpose, and in introducing this chapter we must pause and examine the difference. Arithmetic consists primarily of a rather mechanical application of a large number of rules for computing correctly, and very little attempt is usually made at any real understanding of these rules and of the numbers to which they apply. In algebra, on the other hand, we are interested in understanding rather thoroughly why numbers and the operations we do with them act as they do. We search here for interesting general properties of the numbers and the arithmetic operations with which the student is already familiar. In short, we are interested in what mathematicians sometimes call the "structure" of the "system" of numbers. Other words which convey some of the same meaning as "structure" are "pattern", "form", and "organization".

It is inevitable that many of the general properties of numbers and of the operations we do on them are already quite familiar to the student from the study of arithmetic. The properties are not, however, familiar as explicit principles. Our aim in the last chapter was to have the student discover some of these properties through questions and examples. In this chapter the properties are studied further and formalized. The properties which we have sought to elicit from the student in this way are the commutative and associative properties for both addition and multiplication, the distributive property, and what we call the addition property of 0 and the multiplication property of 1. We have also included the multiplication property of 0, although this could be deduced from the other properties. Later, in Chapters 6 and 7, we see that all these properties of the operations hold for all real numbers. Here we are considering only the non-negative reals.

The student, conditioned as he is to arithmetic, may well say, "Why bother?". We hope to forestall this question through challenging exercises, but the real answer to it is important and consists, to large extent, of what we have just been saying about our concern with structure.

A second goal of this chapter is to develop a good deal of technique in the simplification of algebraic expressions. We do this here because *it is an essential feature of this course that the multitude of exercises required to gain manipulative facility be tied to the ideas from which the techniques derive their validity*. We practice algebraic simplification at the point when the principles upon which such simplification rests are first developed, and many times thereafter. These principles, moreover, are precisely the properties of numbers which the student is to discover in this chapter.

We prepare for the formalization of these properties by first enriching our vocabulary. The concept of a *sentence*, from Chapter 2, is enlarged in three ways: (1) We increase the variety of relations which our sentences can express, so that inequalities are included along with equations, (2) We write *open* sentences which involve variables, and for which the notion of a *truth set* becomes important. It is essential that the student consider both equations and inequalities as sentences, as objects of algebra with equal right to our attention, and as equally interesting and useful types of sentence, (3) We consider compound sentences as well as simple sentences.

While not all of these concepts are immediately necessary for stating the properties of the operations on the numbers of arithmetic, it is worthwhile to introduce them together, and they will be used many times throughout the course.

The teacher may want to read, as a general reference for the work of this chapter, Haag *Studies in Mathematics*, *Volume III Structure of Elementary Algebra*, Chapter 2, Section 2, for the work on sentences; Chapter 1, Section 3 and Chapter 3, Section 2, for background on structure.

Students who have studied the unit on equations in the 8th Grade SMSG material will find that our Chapter 3 is quite a different approach and is considerably more extensive. You will probably want to treat Sections 3-1 through 3-9 as though the students have had little or no former experience with sentences.

The use of variables in Chapter 3 to formulate the properties of addition and multiplication will be familiar to former SMSG students. The applications of these properties, however, are more extensive than those done in the 7th and 8th grades and should be done carefully.

Page 41. The symbols "$=$", "$\neq$", "$<$", "$\geq$", etc. really denote verbs which give a relation between two numbers. Since, however, "is greater than" or "is not equal to" are not single English words, we shall call them "verb forms".

Answers to Problem Set 3-1; Page 41:

1. False
2. True
3. True
4. False
5. True
6. False
7. True
8. False
9. True
10. True
11. True
12. False
13. False
14. True
15. False

Page 42. An open phrase is also "open" in the sense that we cannot determine precisely what number is named by it until we know just what numbers are named by the variables.

Answers to Problem Set 3-2a; Page 43:

1. True when x is 5
2. False when x is 5
3. True when x is 6
4. False when x is 6
5. True when x is 3
   False when x is 4
6. False when x is 4 and y is 3
   True when x is 3 and y is 4
7. False when a is 9 and b is 9
   False when a is 3 and b is 9
8. We do not know until we also know what number m is

In problems 7 and 8, be sure to point out that the given value of "a" must be used in every occurrence of "a".

Page 43. The experimentation with the example "$2x - 11 = 6$" is supposed to suggest a systematic way of guessing values of the variable which will make the sentence true. The method might also suggest how one might decide whether or not all such values have been found. For example, a value of x greater than $8\frac{1}{2}$ will give a number greater than $2(8\frac{1}{2}) - 11$ and a value smaller than $8\frac{1}{2}$ will give a number less than $2(8\frac{1}{2}) - 11$. The properties of order which are suggested here will be taken up later in Chapter 8.

Answers to Problem Set 3-2b; Page 44:

1. 4
2. 10
3. 4
4. $\frac{17}{3}$
5. 2
6. 14
7. no numbers
8. all numbers
9. all numbers
10. no numbers

In Problems 1-6, the student should satisfy himself that all values of the variable have been found by an argument like that described above in which we decide that $\{8\frac{1}{2}\}$ is *the* truth set of $2x - 11 = 6$.

Answers to Problem Set 3-2c; Page 44:

1. (a) 3
   (b) 2
   (c) 1 and 0
   (d) 2
   (e) 7
   (f) 3
   (g) no value
   (h) 0
2. $\frac{3}{4}$
3. 1.4

Page 45. An open sentence involving one variable has a "truth set" defined as the set of numbers for which it is true. We have no need at this time to introduce a name for the set which makes a sentence *false*. The word "solution set" is also used for "truth set", particularly for sentences which are in the form of equations. We shall use "solution set" later, but we want the student to use "truth set" long enough to get its full significance.

Answers to Problem Set 3-3a; Pages 45-46:

1. (a) Since $7 + 5 = 12$ is a true sentence, 5 belongs to the truth set
   (b) Since $6 + 9 \neq 11$ is a true sentence, 6 belongs to the truth set
   (c) Since $\frac{5(3) + 1}{7} \neq 3$ is a true sentence, 3 belongs to the truth set
   (d) Since $2(3) + 1 = 2(3 + 1)$ is a false sentence, 3 does not belong to the truth set
   (e) Since $3 + \frac{1}{3} = 2$ is a false sentence, 3 does not belong to the truth set
   (f) Since $3(5) = 5 + 2(5)$ is a true sentence, 5 belongs to the truth set
   (g) Since $(3)^2 + 2(3) \neq 3(3 + 2)$ is a false sentence, 3 does not belong to the truth set
2. (a) $\{\frac{2}{3}\}$
   (b) $\{1, 3\}$
   (c) $\{1, \frac{1}{6}\}$
   (d) $\{1\}$
   (e) $\{0, 2\}$
   (f) $\{4\}$
   (g) $\{\frac{1}{2}\}$
   (h) $\emptyset$
3. This calls for discussion of at least *several* of the sentences submitted. One possibility is $x = x + 1$.

Page 46. Note that we are careful to read a formula such as $V = \frac{1}{3}Bh$ as a statement about numbers and not about units, square units and cube units. Common usage is not as precise as this but it is important, in the beginning at least, to be sure that the student understands that the operations of addition and multiplication apply to numbers and not to inches, apples, line segments, etc.

Page 47. Problem Set 3-3b. These problems involve important formulas from applications which may or may not be familiar to some of the students. This, however, should not give trouble, since our use of the formulas is independent of the applications. We shall continue to provide experience in working with formulas in later chapters.

Answers to Problem Set 3-3b; Page 47:

1. $C = \frac{5}{9}(86 - 32)$
   $C = \frac{5}{9}(54)$
   $C = 30$
   C is 30
2. $i = prt$
   $120 = 1000(.04)t$
   $120 = 40t$
   t is 3
3. $pv = PV$
   $(15)(600) = 75(V)$
   $(15)(5)(120) = 75(V)$
   $(75)(120) = 75(V)$
   V is 120

*4. $A = \frac{1}{2}(B + b)h$
   $20 = \frac{1}{2}(B + 4)4$
   $20 = \frac{1}{2}(4)(B + 4)$
   $20 = 2(B + 4)$
   B is 6

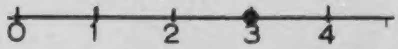
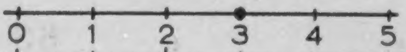
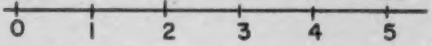
Page 48. We shall soon (page 55) start saying "graph of the open sentence" instead of the more clumsy but more nearly precise "graph of the truth set of the open sentence".

In example (f) do not fuss over the "plotting" of the empty set. Either no graph at all or a number line with no points marked is all right.

For convenience in doing problems involving the number line, you might find it helpful to duplicate sheets of number lines for the pupils use.

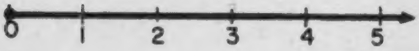
In any of the Problem Sets in which the pupil is asked to find the truth set, see that he checks his results to be sure he has found all numbers which make the sentence true.

Answers to Problem Set 3-4; Page 48:

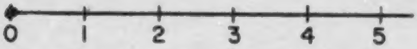
	Truth Set	Graph
1.	{3}	
2.	{3}	
3.	∅	

Truth Set Graph

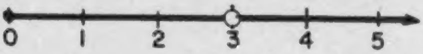
4. all numbers



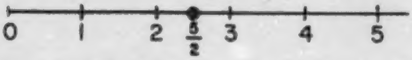
5. $\{0\}$



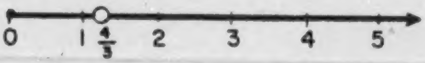
6. all numbers except 3



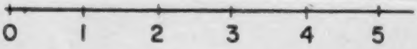
7. $\{\frac{5}{2}\}$



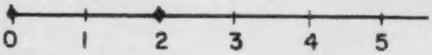
8. all numbers except $\frac{4}{3}$



9. $\emptyset$



10. $\{0, 2\}$



Answers to Problem Set 3-5; Page 49:

1.	False	7.	True
2.	True	8.	False
3.	False	9.	True
4.	True	10.	False
5.	True	11.	False
6.	True	12.	False

Answers to Problem Set 3-6; Pages 50-51:

1. (a) The graph is the graph of the truth set of $2 + x = 4$

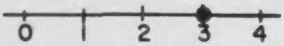
   (b) The graph is not the graph of the truth set of $3x = 5$, because $3(2) = 5$ is false

   (c) The graph is the graph of the truth set of $2y = 7$

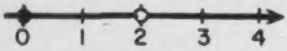
   (d) The graph is not the graph of the truth set of $x > 1$, because the truth set is all numbers greater than 1

   (e) The graph is not the graph of the truth set of $2x > 5$, because $2(\frac{5}{2}) > 5$ is false, hence $\frac{5}{2}$ should not be included

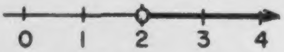
2. (a)



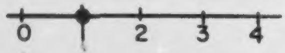
(b)



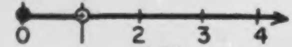
(c)



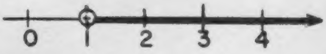
(d)



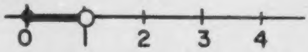
(e)



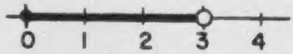
(h)



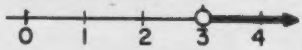
(i)



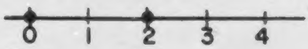
(j)

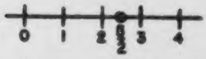


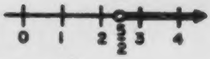
(k)



(l)



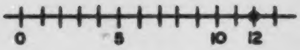
(f) 
0
1
2
$\frac{5}{2}$
3
4

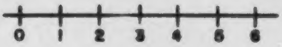
(g) 
0
1
2
$\frac{5}{2}$
3
4

3. (a) $x = 2$

(b) $x < 2$

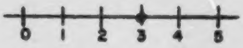
(c) $x \not< 4$

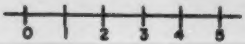
(m) 

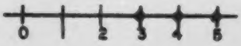
(n) 

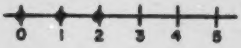
(d) $x > 1$

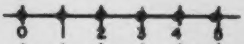
(e) $x \neq 1$

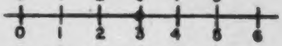
4. (a) $\{3\}$ 

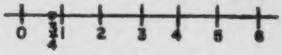
(b) $\emptyset$ 

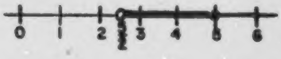
(c) $\{3, 4, 5\}$ 

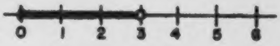
(d) $\{0, 1, 2\}$ 

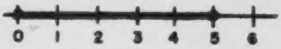
(e) $\{0, 1, 2, 3, 4, 5\}$ 

5. (a) $\{3\}$ 

(b) $\{\frac{3}{4}\}$ 
0
$\frac{3}{4}$
1
2
3
4
5
6

(c) 5 and all numbers greater than $\frac{5}{2}$ and less than 5 
0
1
2
$\frac{5}{2}$
3
4
5
6

(d) all numbers less than 3 

(e) 0, 5, and all numbers greater than 0 and less than 5. 

6. In 4, all truth sets are finite
In 5, the truth sets of the sentences in (a) and (b) are finite

Page 52. We use the word "clause" to denote a sentence which is part of a compound sentence, just as in the corresponding situation in English. The word is convenient but not very important.

Answers to Problem Set 3-7a; Page 52:

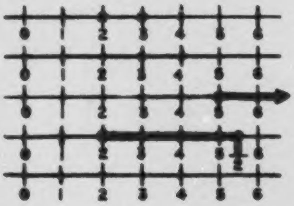
1. True
2. False
3. False
4. True
5. False
6. True

Answers to Problem Set 3-7b; Page 53:

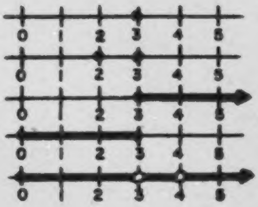
1. True
2. True
3. False
4. False
5. False
6. False

Answers to Problem Set 3-8; Page 55:

1.
2.
3.
4.
5.

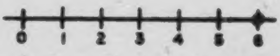

0
1
2
3
4
5
6
$\frac{5}{2}$

6.
7.
8.
9.
10.

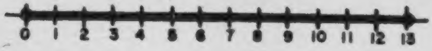


Answers to Problem Set 3-9; Pages 56-57:

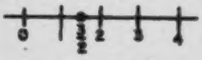
1. $\{6\}$



2. all numbers less than 13

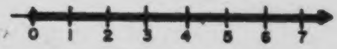

0 1 2 3 4 5 6 7 8 9 10 11 12 13

3. $\{\frac{3}{2}\}$

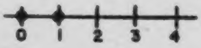

0 1 $\frac{3}{2}$ 2 3 4

4. ∅ no graph

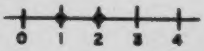
5. all numbers



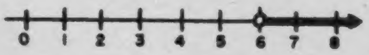
6. $\{0, 1\}$



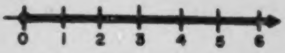
7. $\{1, 2\}$



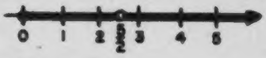
8. all numbers greater than 6



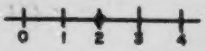
9. all numbers except 0



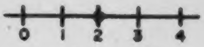
10. all numbers except $\frac{5}{2}$


0 1 2 $\frac{5}{2}$ 3 4 5

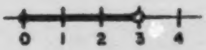
11. $\{2\}$



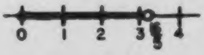
12. $\{2\}$



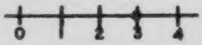
13. all numbers less than 3



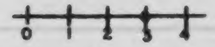
14. all numbers less than $3\frac{1}{5}$



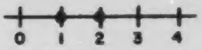
15. {3}



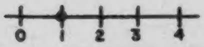
16. {3}



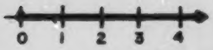
17. {1, 2}



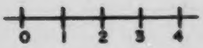
18. {1}



19. all numbers



20. ∅



Page 57. The order of Sections 3-10, 3-11, and 3-12 may seem strange, with the identity elements first, the closure properties second, and the associative and commutative properties third. It was chosen because the properties of 1 and 0 require only one variable in their statement, while the closure properties require two and the associative properties three. Thus the properties which are easiest to state come first.

Page 58. Notice that an open sentence which expresses one of the basic properties is true for all values of the variables. Such sentences convey "structure" or "pattern" information about the number system.

The multiplication property of 0 is not, like the others, a fundamental property of the real number system (it would not, for example, appear among the axioms for an ordered field), but it can be derived from the other properties, and is included here because we shall want the pupil to use it.

Page 60. Problem Set 3-10. If you like, give the students additional instructions something like this: Before you do any really messy computations, take another look at the problem and see if there is an easier way.

Answers to Problem Set 3-10; Page 60:

1. $$\frac{1}{2} + \frac{1}{3} = \frac{1}{2}\left(\frac{3}{3}\right) + \frac{1}{3}\left(\frac{2}{2}\right) = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

2. $$\frac{7}{12} + \frac{5}{18} = \frac{7}{12}\left(\frac{3}{3}\right) + \frac{5}{18}\left(\frac{2}{2}\right) = \frac{21}{36} + \frac{10}{36} = \frac{31}{36}$$

3. $$\frac{7 + \frac{2}{3}}{\frac{5}{6}} = \frac{7 + \frac{2}{3}}{\frac{5}{6}}\left(\frac{6}{6}\right) = \frac{\left(7 + \frac{2}{3}\right)(6)}{\left(\frac{5}{6}\right)(6)} = \frac{42 + 4}{5} = \frac{46}{5}$$

4. $$\frac{\frac{1}{2} + \frac{3}{5}}{\frac{3}{20}} = \frac{\frac{1}{2} + \frac{3}{5}}{\frac{3}{20}}\left(\frac{20}{20}\right) = \frac{\left(\frac{1}{2} + \frac{3}{5}\right)(20)}{\left(\frac{3}{20}\right)(20)} = \frac{10 + 12}{3} = \frac{22}{3}$$

5. $$\begin{aligned} 27 + \left(15\tfrac{2}{3} + 2\tfrac{1}{3}\right) - 18 &= 27 + 18 - 18 \\ &= 27 + 0 \\ &= 27 \end{aligned}$$

6. $$\begin{aligned} \tfrac{7}{8}\left((3.7 + 0.3) - 4\right) &= \tfrac{7}{8}(4 - 4) \\ &= \tfrac{7}{8}(0) \\ &= 0 \end{aligned}$$

7. $$\begin{aligned} 163\left(\tfrac{7}{12} + \tfrac{1}{6} + \tfrac{1}{4}\right) &= 163\left(\tfrac{7}{12} + \tfrac{1}{6}\left(\tfrac{2}{2}\right) + \tfrac{1}{4}\left(\tfrac{3}{3}\right)\right) \\ &= 163\left(\tfrac{7}{12} + \tfrac{2}{12} + \tfrac{3}{12}\right) \\ &= 163\left(\tfrac{12}{12}\right) \\ &= 163 \end{aligned}$$

8. $$\begin{aligned} \left(\tfrac{2}{3} - \tfrac{1}{3}(2)\right) + 17 &= \left(\tfrac{2}{3} - \tfrac{2}{3}\right) + 17 \\ &= 0 + 17 \\ &= 17 \end{aligned}$$

9. $$(6 - 6)(46)(36)\left(17 - \tfrac{2}{3}(12)\right) = 0(46)(36)\left(17 - \tfrac{2}{3}(12)\right)$$

10. $$\begin{aligned} \left(\tfrac{16}{3} - 5\right)(3)(5280) &= \left(\tfrac{1}{3}\right)(3)(5280) \\ &= (1)(5280) \\ &= 5280 \end{aligned}$$

*11. (a) The other must be zero.

(b) At least one of the numbers is 0.

(c) The other property implied here is the converse: if ab = 0, then a is 0 or b is 0.

We return to this property of 0 in Chapter 7.

Page 60. The properties of closure of the set of numbers of arithmetic are implied in the meaning of the operations of addition and multiplication. The student must be reminded over and over again that "a + b" is a numeral (rather than a command to do arithmetic) and reference to the closure property is a convenient way to make the desired point.

[page 60]

Page 62. We enclose "a + b" in parentheses because we are now adding c *to the sum* and hence wish to emphasize that "a + b" is an indicated sum.

What we have done here is translate the English statement of the associative property obtained in Section 2-3 into the language of algebra. This is an example of a type of problem which is considered more systematically in Chapter 4. Some students will find it easy to by-pass the English statement and go directly from the numerical examples to the algebra statement of the property. This could have been done in Section 2-3 except that we did not then have variables and so had to fall back on an English statement. The comparison of the English statement with the algebra statement shows the advantage of the latter in both clarity and simplicity.

Page 62. The associative property for multiplication, stated in the language of algebra, is:

For every number a and every number b and every number c,

$$(ab)c = a(bc).$$

Page 63. In writing phrases in "other forms" we would like to use the expression "simpler form" to describe the end result. Although in most instances it is quite obvious that one form is simpler than another, it appears to be virtually impossible to give a good definition of "simple". Therefore we will be satisfied to use the expression in concrete situations where there is no possibility of confusion and will not attempt to give a general definition. The important idea here is that, when we use the basic properties to write a phrase in another (simpler, more compact, more useful, easier to write, easier to read, better for computation, etc.) form, the result is a phrase which names the same number as the given phrase.

Page 63. The idea here is to emphasize that *there exist* reasonable binary operations which are neither commutative nor associative. (Note that $(12 \div 6) \div 2 = 1$, while $12 \div (6 \div 2) = 4$; and $(2 ** 3) ** 2 = 64$, while $2 ** (3 ** 2) = 512$.) It is felt that examples such as these are needed since it is so *obvious* that the operations of addition and multiplication *do* have the properties. However, *do not spend a lot of time* on these examples (or those in Problems *6 - *8 of Problem Set 3-12) because they are not an end in themselves.

Page 64. Problem Set 3-12. The first two problems particularly prepare for the work with the distributive property. One would like to say "simplest form" in Problem 1.

Problem 3. This problem points up what is now a second approach to truth sets of open sentences. Early in the chapter we guessed such truth sets; now we see that for certain kinds of sentences the truth for all values of the variables involved is a consequence of the fundamental properties of the operations.

Answers to Problem Set 3-12; Pages 64-66:

1. (a) $2m^2n$
   (b) $15p^2q$
   (c) $6n^2m$ (or $6mn^2$)
   (d) $3abc$
   (e) $100ab$
   (f) $36x$

2. (Here we suggest three forms for each. However, there will be other acceptable ones, which can easily be checked by the teacher.)
   (a) $(2a)(4b^2)$
   $(8)(ab^2)$
   $(4ab)(2b)$
   (b) $(7)(xy^2)$
   $(7x)(y^2)$
   $(7xy)(y)$
   (c) $(5m)(2n)$
   $(2m)(5n)$
   $(2mn)(5)$
   (d) $(x)(xy^2)$
   $(xy)(xy)$
   $(x^2y)(y)$
   (e) $(64a)(abc^2)$
   $(4a^2)(16bc^2)$
   $(8abc)(8ac)$
   (f) $(2c)(1)$
   $(2)(c)$
   $(1)(2c)$

3. (a) True for every number m--commutative property of multiplication
   (b) Not true for every number m
   (c) True for every number a, every number b, and every number y
   (d) True for every number x and every number y--commutative property of addition
   (e) Not true for every number a, every number b, and every number c
   (f) True for every number c--commutative property of addition
   (g) True for every number b--commutative property of multiplication
   (h) True for every number n, every number v, and every number z--associative property of multiplication
   (i) True for every number m, every number n, and every number s--associative property of addition
   (j) Not true for every number x and every number y

4. (a) A is not closed under addition
   (b) A is closed under multiplication

5. (a) S is closed under addition
   (b) S is closed under multiplication

*6.

If $a \circ b$ is	$2 \circ 6$ is	$\frac{1}{2} \circ 6$ is	$6 \circ 2$ is	$(3 \circ 2) \circ 4$ is
$2a+b$	$2(2)+6$	$2(\frac{1}{2})+6$	$2(6)+2$	$2(2(3)+2)+4$
$\frac{a+b}{2}$	$\frac{2+6}{2}$	$\frac{\frac{1}{2}+6}{2}$	$\frac{6+2}{2}$	$\frac{\frac{3+2}{2}+4}{2}$
$(a-a)b$	$(2-2)6$	$(\frac{1}{2}-\frac{1}{2})6$	$(6-6)2$	$((3-3)2-(3-3)2)4$
$a+\frac{1}{3}b$	$2+\frac{1}{3}(6)$	$\frac{1}{2}+\frac{1}{3}(3)$	$6+\frac{1}{3}(2)$	$(3+\frac{1}{3}(2))+\frac{1}{3}(4)$
$(a+1)(b+1)$	$(2+1)(6+1)$	$(\frac{1}{2}+1)(6+1)$	$(6+1)(2+1)$	$((3+1)(2+1)+1)(4+1)$

[pages 64-66]

*7. (a) If $a \circ b$ means $\frac{a+b}{2}$, then $b \circ a = \frac{b+a}{2}$.

$a+b = b+a$ commutative property of addition

$$\frac{a+b}{2} = \frac{b+a}{2}$$

$$a \circ b = b \circ a$$

The operation defined here is commutative.

(b) If $a \circ b$ means $(a-a)b$, then $b \circ a$ means $(b-b)a$.

$$0 = 0$$
$$(a-a) = (b-b)$$
$$(a-a)b = (b-b)a$$

The operation defined here is commutative.

(c) If $a \circ b$ means $a + \frac{1}{3}b$, then $b \circ a$ means $b + \frac{1}{3}a$.

Then $3 \circ 6 = 3 + \frac{1}{3}(b)$

$6 \circ 3 = 6 + \frac{1}{3}(3)$

But $3 + \frac{1}{3}(b) = 6 + \frac{1}{3}(3)$ is false, so the operation defined here is not commutative.

(d) If $a \circ b$ means $(a+1)(b+1)$, then $b \circ a$ means $(b+1)(a+1)$.

$(a+1)(b+1) = (b+1)(a+1)$ commutative property of multiplication

$$a \circ b = b \circ a$$

The operation defined here is commutative .

*8. (a) If for every a, and every b, $a \circ b = \frac{a+b}{2}$, is $(4 \circ 2) \circ 5 = 4 \circ (2 \circ 5)$ a true sentence?

$$(4 \circ 2) \circ 5 = \frac{\frac{4+2}{2} + 5}{2}$$
$$= \frac{3 + 5}{2}$$
$$4 \circ (2 \circ 5) = \frac{4 + \frac{2+5}{2}}{2}$$
$$= \frac{4 + \frac{7}{2}}{2}$$
$$= \frac{8 + 7}{4}$$

Since the sentence $\frac{3+5}{2} = \frac{8+7}{4}$ is false, the operation is not associative.

(b) If for every a and every b, $a \bullet b = (a-a)b$, is $(4\circ 2)\circ 5 = 4\circ(2\circ 5)$ a true sentence?

$$\begin{aligned} (4\circ 2)\circ 5 &= \big((4-4)2 - (4-4)2\big)5 \\ &= 0 \\ 4\circ(2\circ 5) &= (4-4)\big((2-2)5\big) \\ &= 0 \end{aligned}$$

Since $0 = 0$, the operation is associative for these particular numbers.

We further note that

$$\begin{aligned} (a\circ b)\circ c &= \big((a-a)b - (a-a)b\big)c \\ &= 0 \\ a\circ(b\circ c) &= (a-a)\big((b-b)c\big) \\ &= 0 \end{aligned}$$

So we conclude that the operation is associative.

(c) If for every a and every b, $a \bullet b = a + \frac{1}{3}b$, is $(4\bullet 2)\circ 5 = 4\circ(2\circ 5)$ a true sentence?

$$\begin{aligned} (4\circ 2)\circ 5 &= \left(4 + \frac{2}{3}\right) + \frac{5}{3} \\ &= \frac{19}{3} \\ 4\circ(2\circ 5) &= 4 + \left(\frac{1}{3}\right)\left(2 + \frac{5}{3}\right) \\ &= 4 + \frac{1}{3}\left(\frac{11}{3}\right) \\ &= 4 + \frac{11}{9} \end{aligned}$$

Since $\frac{19}{3} = 4 + \frac{11}{9}$ is false, the operation is not associative.

(d) If for every a and every b, $a\circ b = (a+1)(b+1)$, is $(4\circ 2)\circ 5 = 4\circ(2\circ 5)$ a true sentence?

$$\begin{aligned} (4\circ 2)\circ 5 &= \big((4+1)(2+1)+1\big)(5+1) \\ &= (16)(6) \end{aligned}$$

$$4\circ(2\circ5) = (4+1)\left((2+1)(5+1) + 1\right)$$
$$= (5)(19)$$

Since $(16)(6) = (5)(19)$ is false, the operation is not associative.

Page 66. It should be emphasized that there are not four different distributive properties. The distributive property is the one stated first and the other three properties are simply different forms for the first and are obtained by using the definition of equality and the commutative property for multiplication.

Answers to Problem Set 3-13a; Page 68:

1. (a) $6r + 6s$
   (b) $ba + 3a$
   (c) $x^2 + xz$
   (d) $7x + x^2$
   (e) $48 + 30$
   (f) $ab + b^2$

2. (a) $3(x + y)$
   (b) $a(m + n)$
   (c) $(1 + b)x$
   (d) $\frac{1}{2}(x + y)$
   (e) $(2 + a)a$
   (f) $x(x + y)$

3. (a) $14x + 3x = (14 + 3)x$
   $= 17x$

   (b) $\frac{3}{4}x + \frac{3}{2}x = (\frac{3}{4} + \frac{3}{2})x$
   $= (\frac{3}{4} + \frac{6}{4})x$
   $= \frac{9}{4}x$

   (c) $\frac{2}{3}a + 3b + \frac{1}{3}a = \frac{2}{3}a + \frac{1}{3}a + 3b$
   $= (\frac{2}{3} + \frac{1}{3})a + 3b$
   $= 1a + 3b$
   $= a + 3b$

   (d) $7x + 13y + 2x + 3y = 7x + 2x + 13y + 3y$
   $= (7 + 2)x + (13 + 3)y$
   $= 9x + 16y$

(e) $$\begin{aligned} 4x + 2y + 2 + 3x &= 4x + 3x + 2y + 2 \\ &= (4 + 3)x + 2y + 2 \\ &= 7x + 2y + 2 \end{aligned}$$

(f) $$\begin{aligned} 1.3x + 3.7y + 6.2 + 7.7x &= 1.3x + 7.7x + 3.7y + 6.2 \\ &= (1.3 + 7.7)x + 3.7y + 6.2 \\ &= 9.0x + 3.7y + 6.2 \end{aligned}$$

(g) $2a + \frac{1}{3}b + 5$--no simpler form

Page 69. In the discussion of applications of the distributive property, a liberal use of parentheses will help emphasize which three numerals are actually involved in the property.

Page 69. Example 3. We have not spelled out any details in this example because these details may seem too formal to the student at this stage. We suspect that he will accept this slightly extended version of the distributive property without question. If he does not, here are the details: In the very act of writing down $3(x + y + z)$, we have used the associative property to permit us to omit grouping symbols inside the parentheses. In order to apply the distributive property as we have stated it, however, we have to put the grouping symbols back in. So, the first step becomes

$$3(x + y + z) = 3\big(x + (y + z)\big)$$

or some other grouping if you prefer. Then, by the distributive property applied twice, we have

$$\begin{aligned} &= 3x + 3(y + z) \\ &= 3x + (3y + 3z). \end{aligned}$$

Finally, we apply the associative property once again to permit us to write, without parentheses,

$$= 3x + 3y + 3z.$$

Answers to Problem Set 3-13b; Pages 69-70:

1. (a) $6m + 3pm$
   (b) $2k^2 + 2k$
   (c) $12s + 18r + 42q$
   (d) $2x^2 + x^2y$
   (e) $eh + fh + gh$
   (f) $6p^2q + 6pq^2$

2. (a) is false for every number a and every number b
   (b) is true for every number x and every number y
   (c) is true for every number a, every number b, and every number c
   (d) is not true for every number a, every number b, and every number c
   (e) is not true for every number x
   (f) is true for every number x and every number y

3. (a) $(3u + v)v$ (d) $(1 + 2d)(2c)$
   (b) $7q(p + r)$ (e) $(1 + 2x)(3x)$
   (c) $(1 + x)3x$ (f) $xz(z + 2)$

Page 70. Examples 1 and 2. Actually, there are parentheses in the second line of each example--for instance $(x^2 + 2x) + (3x + 6)$ in example 1--which are then removed by the associative property.

Answers to Problem Set 3-13c; Page 71:

1. $$\begin{aligned}(x + 4)(x + 2) &= (x + 4)x + (x + 4)2\\ &= x^2 + 4x + 2x + 8\\ &= x^2 + (4 + 2)x + 8\\ &= x^2 + 6x + 8\end{aligned}$$

2. $$\begin{aligned}(x + 1)(x + 5) &= (x + 1)x + (x + 1)5\\ &= x^2 + 1x + 5x + 5\\ &= x^2 + (1 + 5)x + 5\\ &= x^2 + 6x + 5\end{aligned}$$

3. $$\begin{aligned}(x + a)(x + 3) &= (x + a)x + (x + a)3\\ &= x^2 + ax + 3x + 3a\end{aligned}$$

4. $$\begin{aligned}(x + 2)(y + 7) &= (x + 2)y + (x + 2)7\\ &= xy + 2y + 7x + 14\end{aligned}$$

5. $$\begin{aligned}(m + n)(m + n) &= (m + n)m + (m + n)n\\ &= m^2 + mn + mn + n^2\\ &= m^2 + 1mn + 1mn + n^2\\ &= m^2 + (1 + 1)mn + n^2\\ &= m^2 + 2mn + n^2\end{aligned}$$

6. $$\begin{aligned}(2p + q)(p + 2q) &= (2p + q)p + (2p + q)2q \\ &= 2p^2 + pq + 4pq + 2q^2 \\ &= 2p^2 + (1 + 4)pq + 2q^2 \\ &= 2p^2 + 5pq + 2q^2\end{aligned}$$

Page 71. This summary of properties is very important. We want the student to begin thinking of the number system more-and-more often in terms of the basic properties so that eventually almost everything that he does with numbers will be done with these properties in mind. This is a development which will not take place for most students very quickly, but, by the end of the year, it is hoped that the majority will have progressed to within sight of the goal.

The list of properties obtained at this point is not complete. We still must introduce the negative numbers and obtain the properties of order. The list will be completed for our purposes by the end of Chapter 8.

Answers to Problem Set 3-14; Page 72:

1. $$\begin{array}{lll} (8 \times 3) + 2 = 26 & 8 + (3 + 2) = 13 & 8 - (3 \times 2) = 2 \\ 8 \times (3 + 2) = 40 & (8 + 3) + 2 = 13 & (8 - 3) \times 2 = 10 \\ (8 \times 3) - 2 = 22 & 8 + (3 \times 2) = 14 & 8 - (3 + 2) = 3 \\ 8 \times (3 - 2) = 8 & (8 + 3) \times 2 = 22 & (8 - 3) + 2 = 7 \\ 8 \times (3 \times 2) = 48 & 8 + (3 - 2) = 9 & 8 - (3 - 2) = 7 \\ (8 \times 3) \times 2 = 48 & (8 + 3) - 2 = 9 & (8 - 3) - 2 = 3 \end{array}$$

This is an interesting study in arrangements. The 8, 3, and 2 are fixed. The first of the signs may be ×, +, or -, and for each of these the second of the signs may be ×, + or -. Then there are two ways in which parentheses may be inserted, grouping either the first two terms or the last two. After all this is done, it is interesting to note which expressions are names for the same number. For instance:

$8 - (3 + 2) = (8 - 3) - 2$, and in Chapter 7 we show why
$8 + (3 + 2) = (8 + 3) + 2$ this is so.
$8 \times (3 \times 2) = (8 \times 3) \times 2$

2. (a) $5 + 5x$

(b) $\frac{1}{4}a(3 + 2)$ or $a(\frac{3}{4} + \frac{2}{4})$

(c) $11[3(95) + 15]$
or $5[33(19) + 3(11)]$
or $15[11(19) + 11]$
or $165(19 + 1)$

(d) $(1 + 2)c$

(e) $\frac{1}{2}y + \frac{1}{3}y$

(f) $xy + x$

(g) $x(y + 1)$

(h) $a(3x + 2y)$

(i) $(5 + y + 2)y$
or $(7 + y)y$

(j) $(2 + a)(b + 3)$

(k) $(1 + 3a)b$

(l) $2a^2 + 2ab + 2ac$

(m) $(u + 2v)u + (u + 2v)v$
$= u^2 + 2uv + uv + 2v^2$
$= u^2 + 3uv + 2v^2$

(n) $(a + 1)a + (a + 1)1$
$= a^2 + 1a + 1a + 1$
$= a^2 + 2a + 1$

3. (a) $17x + x = (17 + 1)x$
$= 18x$

(b) $2x + y + 3x + y = 2x + 3x + 1y + 1y$
$= (2 + 3)x + (1 + 1)y$
$= 5x + 2y$

(c) $3(x + 1) + 2x + 7 = 3x + 3 + 2x + 7$
$= 3x + 2x + 3 + 7$
$= (3 + 2)x + 10$
$= 5x + 10$

(d) $1.6a + .7 + .4a + .3b = 1.6a + .4a + .7 + .3b$
$= 2.0a + .7 + .3b$

(e) $by + 2by = (1 + 2)by$
$= 3by$

(f) $9x + 3 + x + 2 + 11x = 9x + 1x + 11x + 3 + 2$
$= (9 + 1 + 11)x + 5$
$= 21x + 5$

*4. Since $(2 + 3 + 5 + 7)$ is not divisible by 9, we cannot use the distributive property to write 2357 as the product of 9 and a whole number, so we find that 2357 is not divisible by 9.

$$\begin{aligned}
35874 &= 3(10{,}000) + 5(1000) + 8(100) + 7(10) + 4(1) \\
&= 3(9999+1) + 5(999+1) + 8(99+1) + 7(9+1) + 4(1) \\
&= 3(9999) + 3(1) + 5(999) + 5(1) + 8(99) + 8(1) + \\
&\quad 7(9) + 7(1) + 4(1) \\
&= \big(3(9999) + 5(999) + 8(99) + 7(9)\big) + 3(1) + 5(1) + \\
&\quad 8(1) + 7(1) + 4(1) \\
&= \big(3(1111) + 5(111) + 8(11) + 7(1)\big)9 + (3 + 5 + 8 + \\
&\quad 7 + 4) \\
&= (3333 + 555 + 88 + 7)9 + (27) \\
&= (3333 + 555 + 88 + 7 + 3)9
\end{aligned}$$

Hence 35874 is divisible by 9

The rule which becomes apparent is that if the sum of the digits of a number is divisible by 9, the number is divisible by 9.

*5. $19 \times 13 = 19(10 + 3)$

$= 19(10) + 19(3)$ - distributive property

$= 19(10) + (10 + 9)3$

$= 19(10) + \big(10(3) + 9(3)\big)$ - distributive property

$= \big(19(10) + 10(3)\big) + 9(3)$ - associative property of addition

$= (19 + 3)10 + 9(3)$ - distributive property

$15 \times 14 = (15 + 4)10 + 20$

$13 \times 17 = (13 + 7)10 + 21$

$11 \times 12 = (11 + 2)10 + 2$

Answers to Review Problems; Pages 73-75:

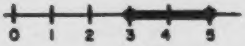
1. (a) H is the set of odd integers greater than 19 and less than 51 or the set of odd integers from 21 to 49 inclusive.

   (b) H is a subset of A, but A is not a subset of H.

   (c) Set H is finite, Set A is infinite.

2. Since $\frac{3}{4}$ is $\frac{18}{24}$ and $\frac{5}{6}$ is $\frac{20}{24}$, the coordinate of one point between the two is $\frac{19}{24}$. There are infinitely many points between $\frac{3}{4}$ and $\frac{5}{6}$.

3. Set T is closed under addition, since the sum of two elements gives an integral multiple of 3. Set T is not closed under "averaging", since the average of two elements (such as 3 and 6) is not necessarily an integer.

4. (a) 

   (b) $\frac{3}{2}$ is not an admissible value of t

   π is an admissible value of t

   $\frac{17}{4}$ is an admissible value of t

   $4\frac{1}{3}$ is an admissible value of t

   $\frac{11}{2}$ is not an admissible value of t

   $\sqrt{2}$ is not an admissible value of t

5.
$$\begin{aligned}(3 + 1)^2 &= (3 + 1)(3 + 1)\\ &= (3 + 1)3 + (3 + 1)1\\ &= 3^2 + (3)(1) + (3)(1) + 1^2\\ &= 3^2 + 2(3)(1) + 1^2\end{aligned}$$

$$\begin{aligned}(5 + 2)^2 &= (5 + 2)(5 + 2)\\ &= (5 + 2)5 + (5 + 2)2\\ &= 5^2 + (5)(2) + (5)(2) + 2^2\\ &= 5^2 + 2(5)(2) + 2^2\end{aligned}$$

$$\begin{aligned}(\tfrac{1}{3} + \tfrac{2}{3})^2 &= (\tfrac{1}{3} + \tfrac{2}{3})^2\\ &= (\tfrac{1}{3} + \tfrac{2}{3})(\tfrac{1}{3}) + (\tfrac{1}{3} + \tfrac{2}{3})(\tfrac{2}{3})\\ &= (\tfrac{1}{3})^2 + (\tfrac{1}{3})(\tfrac{2}{3}) + (\tfrac{1}{3})(\tfrac{2}{3}) + (\tfrac{2}{3})^2\\ &= (\tfrac{1}{3})^2 + 2(\tfrac{1}{3})(\tfrac{2}{3}) + (\tfrac{2}{3})^2\end{aligned}$$

For every a and every b,

$$(a + b)^2 = a^2 + 2ab + b^2$$

Proof:

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b)\\ &= (a + b)a + (a + b)b\\ &= a^2 + (b)(a) + (a)(b) + b^2\\ &= a^2 + (1)(a)(b) + (1)(a)(b) + b^2\\ &= a^2 + (1 + 1)(a)(b) + b^2\\ &= a^2 + 2ab + b^2\end{aligned}$$

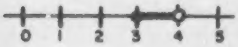
6. (c)

7. (a) $\{12\}$
   (b) $\{6\}$
   (c) $\{4\}$
   (d) $\{3\}$
   (e) $\{2\}$
   (f) $\{1\}$

8. If m is a number of arithmetic, the truth sets are:
   (a) $\{1\}$
   (b) The set of all numbers of arithmetic
   (c) $\{0\}$
   (d) $\emptyset$

   The truth set of (d) is a subset of all the others.
   The truth set of (b) is a subset of none other than itself.
   If the domain of m is the set of counting numbers, the truth sets are:
   (a) $\{1\}$
   (b) The set of all counting numbers
   (c) $\emptyset$
   (d) $\emptyset$

9. (a) 3 is an element of T
   (b) 2 is an element of T
   (c) $\emptyset$ is a subset of T ($\emptyset$ is a subset of every set)

10. 

11. (a) $\emptyset$
    (b) $\{0\}$
    (c) 0 and $\frac{1}{2}$ and all numbers between 0 and $\frac{1}{2}$

12. (a) The sentence is true.
    (b) All that is needed is to note that $(8 + 1)$ and $(4 + 5)$ are names for the same number.

13. (a) True (d) True
    (b) False (e) True
    (c) True (f) True

14.
$$\frac{\frac{3}{5} + \frac{2}{3}}{\frac{3}{4}} = \frac{\frac{3}{5} + \frac{2}{3}}{\frac{3}{4}}\left(\frac{60}{60}\right)$$
$$= \frac{\left(\frac{3}{5} + \frac{2}{3}\right)(60)}{\left(\frac{3}{4}\right)(60)}$$
$$= \frac{36 + 40}{45}$$
$$= \frac{76}{45}$$

15. $3x + y + 2x + 3y = (3x + 2x) + (y + 3y)$ by the associative and commutative properties of addition
$= (3 + 2)x + (1 + 3)y$ by the distributive property
$= 5x + 4y$

Since the associative, commutative, and distributive properties are true for all numbers,

$$3x + y + 2x + 3y = 5x + 4y$$

for all numbers.

16. (a)
$$\begin{aligned}(x + 1)(x + 1) &= (x + 1)x + (x + 1)1 &&\text{- distributive}\\ &= x^2 + x + x + 1 &&\text{- distributive}\\ &= x^2 + (1 + 1)x + 1 &&\text{- distributive}\\ &= x^2 + 2x + 1\end{aligned}$$

$$(x + 2)(x + 2) = x^2 + 4x + 4 \text{ - as above}$$

(b)
$$\begin{aligned}x^2 + 6x + 9 &= x^2 + (3 + 3)x + 9\\ &= x^2 + 3x + 3x + 9 &&\text{- distributive}\\ &= (x + 3)x + (x + 3)3 &&\text{- distributive}\end{aligned}$$

Since $x + 3$ is a number
by closure

$$= (x + 3)(x + 3) \text{ - distributive}$$

The student is more likely to guess that since $(x + 1)(x + 1)$ and $(x + 2)(x + 2)$ were the first two products, $(x + 3)(x + 3)$ would be a likely product to get $x^2 + 6x + 9$. If he demonstrates that $(x + 3)(x + 3) = x^2 + 6x + 9$, we would consider this a proper procedure.

Chapter 3

Suggested Test Items

1. Which of the following sentences are true and which are false?

   (a) $5 \leq 6 + 1$

   (b) $9 + 11 \ngtr 5$

   (c) $8 + 2 > 3$

   (d) $5 + 3 \neq 7$

2. Which of the following sentences are true and which are false?

   (a) $\frac{6 - 3}{3} > \frac{3}{2}$ and $6 \ngtr 2 - 1$

   (b) $6 \neq 5 + 1$ or $4 \leq 3$

   (c) $6 + 1 = 4 + 3$ or $6 \nless 7$

   (d) $\frac{4}{7} + \frac{3}{4} > \frac{3}{5} + \frac{5}{7}$ and $4 \nless 5.3$

3. Which of the following sentences are true and which are false?

   (a) $(18 - 10) - 4 = 18 - (10 - 4)$

   (b) $(18 - 10) - 4 \neq 18 - (10 + 4)$

   (c) $3 + 4 < 8$ or $6 + 5 > 5 + 6$

   (d) $7 + 0 = 7$ and $7(0) = 7$

   (e) $4 > 6$ or $5 + 2 = 10$

   (f) $7 \not< 3$ or $17.813 + .529 = 8.777 + 18.442$

4. Determine whether each sentence is true for the given value of the variable.

   (a) $3t + 4 = 15$; 2
   (c) $20 - 2x \not< 10$; 5

   (b) $4x - 3 < 7$; 7
   (d) $\frac{x}{2} + \frac{1}{3} \neq \frac{6x}{12} + \frac{3}{9}$; 1

5. If the variables have the values assigned below, determine whether the sentence is true.

   (a) $3x = 4 + y$, x is 2 and y is 2

   (b) $5x < 2 + y$, x is 3 and y is 8

6. List the truth set of each of the following open sentences. Use the suggested trial numbers to help discover each truth set.

   (a) $x + 3 = 3x - 5$; $\{2, 4, 6\}$

   (b) $x^2 + 3 = 4x$; $\{0, 1, 2, 3\}$

7. Determine the truth sets of the following open sentences:

   (a) $3x + 4 = 25$
   (c) $2x + 3 = 2x + 5$

   (b) $2x + 1 < 3$
   (d) $4 + x \not> 2x + 1$

8. Draw the graphs of the truth sets of the open sentences:

   (a) $x + 5 = 6$
   (c) $2x \leq 7$

   (b) $x + 1 > 3$
   (d) $x \not> 4$

9. Draw the graphs of the truth sets of the compound open sentences:

(a) $x > 3$ and $x < 4$ (c) $x = 5$ or $x < 4$

(b) $x \leq 5$ and $x > 4$ (d) $x < 3$ and $x > 4$

10. The following condition is required of a number called n: "if the product of 6 and the number n is increased by 4, the result is 40."

(a) Write the condition as an open sentence.

(b) What is its truth set?

(c) Graph the truth set.

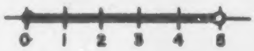
11. Which of the open sentences A, B, C, D, and E below has the same truth set as the open sentence "$p \geq 7$"?

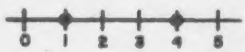
A. $p > 7$ or $p = 7$ D. $p \not< 7$

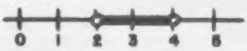
B. $p > 7$ and $p = 7$ E. $p \neq 7$

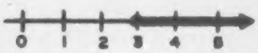
C. $p \not> 7$

12. Write open sentences whose truth sets are the sets graphed below:

(a)


(c)


(b)


(d)


13. For each of the sentences in column I, select the appropriate graph of its truth set in column II.

I

(a) $6x = 18$

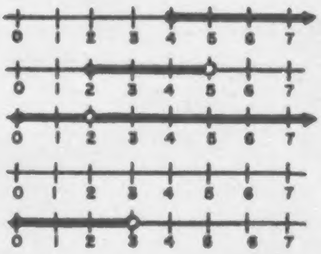
(b) $y < 3$

(c) $b \neq 2$

(d) $t > 4$

(e) $d \geq 2$ and $d < 5$

II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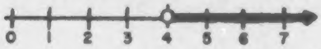
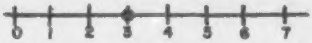
(f) $x \geq 4$



(g) $w < 2$ and $w > 4$



14. If the domain of the variable is the set $U = \{2, 4, 6, 8, 10, 12\}$, find truth sets for the following open sentences:

(a) $3x + 1 = 13$ (c) $2x < 20$ and $x + 4 = 4 + x$

(b) $2x = 10$ (d) $2x + 1 = 7$ or $2x - 1 = 3$

15. The formula used to change a temperature C measured in degrees Centigrade to the corresponding temperature F in degrees Fahrenheit is

$$F = \frac{9}{5}C + 32 .$$

Find the value of F when C is 90.

16. The formula for the area of a triangle is

$$A = \frac{1}{2}bh,$$

where A is the number of square units in the area, b is the number of units in the base, and h is the number of units in the height. Find the value of h when A is 35 and b is 10.

17. Each sentence below is true for every value of the variables. In each case tell which properties are involved in showing that the sentence is true.

(a) $x(y + z) = xy + xz$ (e) $(ab)(cd) = (dc)(ba)$

(b) $xy + (ay + c) = (xy + ay) + c$ (f) $x + 0 = x$

(c) $abcd = ab(cd)$ (g) $0(x) = 0$

(d) $xy + xz = yx + zx$ (h) $1(x) = x$

18. Show how you use the multiplication property of 1 to find common names for:

(a) $\frac{2}{3} + \frac{3}{4}$ (b) $\frac{2 + \frac{1}{3}}{5}$

19. Tell which of the following sentences are true for every value of the variable:

(a) $x(2 + 3) = (2 + 3)x$

(b) $b(a + 2) = a(b + 2)$

(c) $a^2b^2 = b^2a^2$

(d) $(4x + y)3 = 4x(y + 3)$

(e) $(3a + c) + d = (c + d) + 3a$

(f) $(2x)y = 2(xy)$

(g) $a(b - b) = a$

(h) $a + (b - b) = a$

20. Show how you might use the distributive property to do the following computations more easily:

(a) $(324)(102)$

(b) $(37)(2\frac{2}{3}) + (37)(3\frac{1}{3})$

(c) $60(\frac{5}{6} + \frac{3}{5})$

(d) $(13)(29)$

21. For each step in the following sequence state which property of operations was used to derive it from the preceding step:

$$\begin{aligned} 3(2x + y) + 5x &= (6x + 3y) + 5x \\ &= (3y + 6x) + 5x \\ &= 3y + (6x + 5x) \\ &= 3y + (6 + 5)x \\ &= 3y + 11x \end{aligned}$$

22. Use the properties of operations to write the following open phrases in simpler form:

(a) $6x + 3x$

(b) $\frac{5}{4} + \frac{2}{7}x + 3\frac{1}{5}y + \frac{5}{7}x + \frac{4}{5}y$

(c) $414a + 37b + 86a + 63b$

(d) $3x + 4y + 7x + z + 2y$

23. Write each of the following open phrases as an indicated product:

(a) $2x + xy$

(b) $(2p + q)u + (2p + q)v$

(c) $7(x + 2yz) + xy(x + 2yz)$

(d) $y(x + 1) + 7y$

(e) $a(b + 3) + 3(b + 3)$

24. Write each of the following indicated products as an indicated sum without parentheses:

(a) $7a(3b + 2c)$

(b) $5x(2x + 3y)$

(c) $(2a + 3b)(4c + 5d)$

(d) $(a + b)^2$

(e) $(3x + 2y)(3x + 2y)$

25. Show that

$$ab + (cb + st) = b(a + c) + st$$

is true for all values of the variables. (Write out the process step by step, giving the property used in each step.)

*26. If for every number a and every number b, $a \circ b$ means $a + 3b$, answer the following questions:

(a) State values for: $3 \circ 4$; $2 \circ (3 \circ 4)$; $(2 \circ 3) \circ 4$; $4 \circ 3$.

(b) Is the operation commutative?

(c) Is the operation associative?

Chapter 4

OPEN SENTENCES AND ENGLISH SENTENCES

The purpose of this chapter is to help develop some ability in writing open sentences for word problems. We work first just with phrases. We do some inventing of English phrases to fit open phrases at the start to try to help give a more complete picture of what this translation back and forth is like. Then we translate back and forth sentences involving both equations and inequalities.

In order to concentrate on the translation process, we prefer at present not to become involved in finding truth sets of the open sentences. We nevertheless have asked questions in the word problems. This seems necessary in order to point clearly to a variable, to make the experience more closely related to the problem solving to which this translation process will be applied, and to bring forth the sentence or sentences we are really wanting the student to think of. Thus in Example 1 on page 82 if instead of saying "How long should each piece be?" we said "Write an open sentence about the lengths of the pieces.", the student might well answer "If one piece is n inches long, then $n < 44$," or he might even answer "$n > 0$". These are true enough sentences, but they miss the experience we want the students to have.

Some students may feel the urge to go on and "find the answer." In that case you should let them try, but don't let "finding the answer" become a distraction at this point. Tell the students that we will be developing more efficient methods of finding truth sets of sentences later on, but for the present we shall go no further than writing the open sentence.

In a few of the problems in this course there is superfluous information which is not necessary for doing the problem. This is intentional. We hope that occasional experience with such irrelevant material will help make the student more aware of information which _is_ relevant.

4-1. Open Phrases and English Phrases

Page 77. In translating from open phrases to English phrases, many English phrases are possible. Encourage the students to use their imaginations and bring in as great a variety of translations as possible.

Possible translations of "-" include "less than", "the difference of", "shorter than", etc. You may have to warn your students that, since subtraction is not commutative, they must watch which number comes first in using "less than".

You will sooner or later find a student who is confused about the difference between "greater than" or "more than" which calls for "+", and "is greater than" or "is more than" which calls for ">". Be prepared to make this distinction clear. Thus, "This turkey weighs five pounds more than that one" could call for the phrase "t + 5", while "This turkey weighs more than twenty pounds," could call for the sentence "p > 20".

Note carefully in the following problems that in describing the variable, w is not "width" but "number of feet in the width," x is not "books" but "the number of books Mary has." b is not "the boy" but "the number of years in the boy's age." Notice also that a clear, correct, and smoothly flowing way to say the last phrase is "the boy is b years old."

Some of the problems in this chapter may involve more than one variable or may suggest the use of more than one variable. This should be allowed to happen casually. In the case of open sentences you may have opportunity to show the possibility of a compound sentence. It is too early to be able to show the necessity of a compound sentence for a unique solution. Since we are not at present looking for answers it will not be necessary to worry yet about how we will find the truth set.

Answers to Problem Set 4-1; pages 78-81:

The English phrases in Problems 1-6 and 18-25 are suggestions only. As much variety as possible on the part of the class should be encouraged.

For many of the following problems there are implied restrictions on the domain of the variable. While we ordinarily let such restrictions remain implied because they seem quite obvious, there would be some value in occasionally discussing with the students what restrictions actually exist.

For instance, in this Problem Set the domain is limited to whole numbers in Problems 11, 14, 15, 30, 31, 34, and some of the suggested translations from open phrases to English phrases given below impose the same limitation on the domain. When the variable represents a number of dollars, the domain cannot include numbers like $\frac{3}{7}$. In Problem 12, n cannot be zero. In Problem 31 there is probably a further limitation on the domain of the variable imposed by the capacity of the auditorium.

Of course, at this point in the course we are restricting ourselves to the numbers of arithmetic, but most of the problems by their nature give only non-negative numbers in the domain anyway.

1. See text.
2. If Jonathan weighs n pounds, the phrase is: The number of pounds in the weight of a boy who is 7 pounds heavier than Jonathan.
3. If I am n years old now, the phrase is: The number of years in my age 7 years ago.
4. If there are y students in the class, the phrase is: Half of the students in the class.
5. If rhubarb costs r cents per bunch and a shopping bag costs 5 cents, the phrase is: The cost in cents of 2 bunches of rhubarb and a shopping bag to put them in.
6. If a coal mine produced a tons of coal one day and b tons of coal the next day, the phrase is: The number of tons of coal produced in the two days. (Notice that we are using two variables here. While it will be some time before the students do much with more than one variable, we need not shut our eyes to them.)

[pages 78-81]

7. See text

8. If f is the number of feet, then the phrase is 12f.

9. If k is the number of quarts, then the phrase is 2k.

10. If k is the number of feet, then the phrase is $\frac{k}{5280}$.

11. If n is the whole number, then the phrase is n + 1.

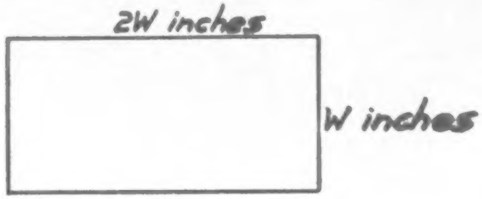
12. If n is the number, then the phrase is $\frac{1}{n}$.

13. If there are k pounds and t ounces, then the phrase is 16k + t.

14. If there are d dollars and k quarters, then the phrase is 100d + 25k.

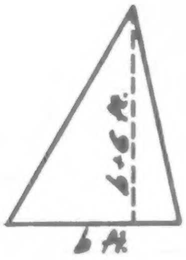
15. If there are m dollars, k quarters, m dimes, and n nickels, then the phrase is 100m + 25k + 10m + 5n. (Notice that m represents the number of dimes as well as the number of dollars. This means that we have the same number of dimes as dollars, since a variable cannot represent two different numbers at the same instant. The open phrase might be written 110m + 25k + 5n.)

16.



If the rectangle is w inches wide, then the phrase is 2w.

17. If the base of the triangle is b feet long, then the phrase is b + 6.



18. If I bought n bracelets, then the phrase is: The total cost in dollars of the bracelets I bought at \$1 each and the same number of necklaces at \$7 each.

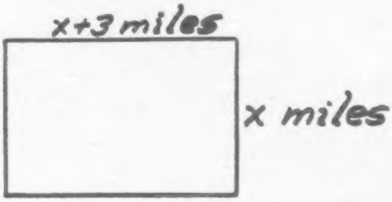
19. If there are r paper clips in each box, then the phrase is: The number of paper clips I now have if I had two boxes of them and my roommate took away five paper clips and then brought back seven.

20. If the first side of a triangle is x inches long, then the phrase is: The number of inches in the perimeter of a triangle if the second side is three times the first one and the third side is one inch longer than twice the first side.

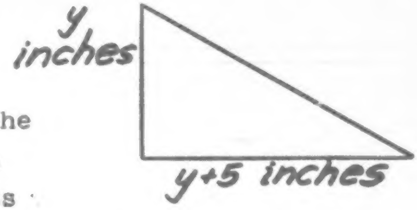
21. If y is the number of people in each of the new families, then the phrase is: The population of a village of 5,000 people after four more families move in.

22. If I put y dollars in the bank, then the phrase is: The number of dollars in the bank after simple interest for one year at 4% has been added.

*23. If a rectangular National Forest is x miles wide, then the phrase is: The number of square miles in the area of the forest if it is 3 miles longer than it is wide.



*24. If the altitude of a triangle is y inches, then the phrase is: The number of square inches in the area of the triangle if its base is 5 inches longer than its altitude.



*25. If a room is w feet high, then the phrase is: The number of cubic feet in the volume of the room if its width is 3 feet more than its height and its length is 5 feet more than twice the height.

26. If one side of the square is a feet long, then the phrase is $4a$.

27. If a man's head is h inches high, then the phrase is $(7\frac{1}{2})h$ or $\frac{15}{2}h$.

28. If the first side of a triangle is f inches long, then the phrase is $f + 3$.

29. (a) $x + y + \frac{1}{2}(x + y)$
(b) $\frac{1}{2}(x + y)$

30. If there are b boys, then the phrase is $\frac{6}{5}b$. The students may need a little help on the words "in terms of." You can explain that this is another, and a frequently used way of pointing to the variable.

31. If a people bought tickets, then the phrase is $2a$.

32. $\frac{1}{d}$ An inductive approach may help the student to see some of these. Here he will probably respond to the question, "How much of the house will he paint in one day if it takes him 8 days to paint the house?"

33. $x \cdot \frac{1}{5}$ or $\frac{x}{5}$

Note: Help the student to notice that when the variable is given in the problem, it is not necessary for him to tell about it, but if the problem does not give the variable, it is the student's responsibility to choose a letter and tell what it represents.

34. $r + 12$

*35. If the plant grows g inches per week, then the phrase is $20 + 5g$.

*36. If t is the number of minutes after his arms are immersed, and $t \geq 10$, then the phrase is $t - 10$.

37. If x is the number of dollars in the inheritance, then the phrase is:

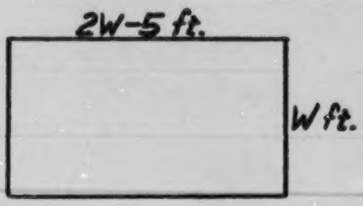
(a) $\frac{1}{2}x$ or $\frac{x}{2}$

(b) $\frac{1}{10}x + 50$ or $\frac{x}{10} + 50$

(c) $x - \left(\frac{1}{2}x + \left(\frac{1}{10}x + 50\right)\right)$ or $x - \left(\frac{x}{2} + \left(\frac{x}{10} + 50\right)\right)$

(d) x

Of course in (d) the student might write $\left(\frac{1}{2}x\right) + \left(\frac{1}{10}x + 50\right) + \left(x - \left(\frac{1}{2}x + \left(\frac{1}{10}x + 50\right)\right)\right)$. He might have fun showing that this phrase represents the same number as x.



38. If the rectangle is w feet wide, then the phrase is

(a) $2w - 5$

(b) $w + (2w - 5) + w + (2w - 5)$
or $2w + 2(2w - 5)$

(c) $w(2w - 5)$

In the early part of our work with translation we have been trying to emphasize the idea that the variable represents a number by being reasonably precise in the language. Thus we have been saying, "the number of feet in the width", "the number of inches in the base", etc. As we go on, we become more careless about this way of speaking in order to be able to speak more fluently. So in Problem 38, as long as the unit of length and the unit of area are clearly stated elsewhere in the problem, we allow ourselves to speak of "perimeter" and "area" instead of "number of feet in the perimeter" and "number of square feet in the area."

4-2. Open Sentences and English Sentences

We give suggested translations for the open sentences. Assigning them to the students will produce a great variety of translations. One way of testing the correctness of the student translations might be to distribute them about the class and have the students try translating them back into open sentences. This would also serve to give the students a start on the following work by having them first translate student-made English sentences into open sentences.

Answers to Problem Set 4-2a; page 82:

1. Sammy, who is 7 pounds heavier than Jonathan, weighs 82 pounds.

2. In order to buy 500 envelopes, I had to buy two boxes of envelopes.

3. Seventeen of the students did supplementary projects. This is half the class.

4. A rectangular field is 4 rods longer than it is wide and its area is 480 square rods.

5. I have three pieces of chain. The second piece has twice as many links as the first and the third piece has three times as many links as the first. I get a single chain with the same number of links whether I fasten the second and third pieces together and then fasten them to the first, or whether I fasten the first and second pieces together and then fasten the third to them.

6. James bought some 4 cent stamps and the same number of 7 cent stamps. The total cost was 44 cents.

7. A contractor ordered 4 bags of nails and later ordered 7 more bags of the same kind of nails. The total shipping weight was 47 pounds.
   Note: The difference between Problem 6 and Problem 7 is that in 6 you may restrict the domain of the variable to whole numbers and there will be a whole number, namely

4, which will make the sentence true. In Problem 7 this is not true, so it would be incorrect to set up a sentence which requires its variable to represent a whole number. *If at all possible, let the students discover this fact themselves.*

8. The perimeter of a square is 100 feet.

9. I have to travel just as far whether I go from here to Fairwood and then the five miles to Middlebury, or whether I go first from Middlebury to Fairwood and then back here.

10. We received 6 gallons of milk, some in quart bottles and some in half-gallon bottles.

Page 82. There are several important points to be noted here.
1. What question is the problem asking? The answer to this question should be stated in terms of a *number*. Do not allow the student to write, "x = the cost" or "y = the length." Help him to become conscious of the fact that the variable represents a *number*. He should say, "x is the number of dollars in the cost of the house", or more smoothly, "the house costs x dollars."
2. Any *other* numbers needed in the problems should be stated in terms of the one named by the variable. Thus we say, "If the shorter piece is x inches long, the longer piece is (x + 3) inches long." Of course some situations may naturally lend themselves to the use of two variables. As we have said before, (page 76) there is no objection in this chapter to including an occasional example of this sort.
3. There should be a direct translation into an open sentence. Thus for Example 1 on page 82 while we *could* change the sentence to $2K + 3 = 44$, such a sentence is not a direct translation of the problem. It does not really tell the story. A good test of a direct translation is to see whether, with the description of the variable, the sentence can be translated readily back into the original problem.

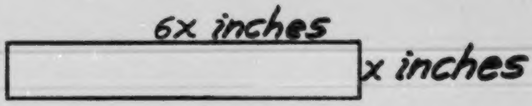
4. We repeat that, for the present, the emphasis is on the translating and we do not expect the students to go on and "get the answer" in this chapter. You should be warned that Examples 1, 2, and 3 on pages 82-84 do not have obvious answers. This was so planned in order to try to avoid the distraction of "seeing the answers."

Answers to Problem Set 4-2b; pages 84-86:

1. If Charles received c votes, then Henry received $c + 30$ votes, and $c + (c + 30) = 516$.

   If Charles received c votes, then Henry received $516 - c$ votes, and $516 - c = c + 30$.

2. If the rectangle is x inches wide, then it is $6x$ inches long, and $x + 6x + x + 6x = 144$ or $2x + 2(6x) = 144$.



3. If the smallest angle is s degrees, then the largest angle is $2s + 20$ degrees, and $s + (2s + 20) + 70 = 180$.

4. If y is the number of feet in one of the shorter spans, then the long span is $y + 100$ feet long, and $y + y + (y + 100) = 2500$.

5. If there were y students in Miss Jones's class, then there were $y + 5$ students in Mr. Smith's class and $y + (y + 5) = 43$.

   If there were y students in Miss Jones's class, then there were $43 - y$ students in Mr. Smith's class and $43 - y = y + 5$.

   (Call attention to the second of these methods. It is a useful approach to know, but one with which some students have trouble at first.)

6. If the rectangle is w inches wide, then it is $w + 5$ inches long and $w(w + 5) = 594$.

7. If Dick is x years old now, then John is $3x$ years old now. Dick was $x - 3$ years old three years ago, John was $3x - 3$ years old three years ago, and $(x - 3) + (3x - 3) = 22$.

8. If John has d dimes, then he has $d + 1$ quarters, he has $2d + 1$ nickels, his dimes are worth $10d$ cents, his quarters are worth $25(d + 1)$ cents, his nickels are worth $5(2d + 1)$ cents and $10d + 25(d + 1) + 5(2d + 1) = 165$.

9. If I bought f four-cent stamps, then I bought $23-f$ seven-cent stamps, the four-cent stamps cost $4f$ cents, the seven-cent stamps cost $7(23 - f)$ cents, and $4f + 7(23 - f) = 119$.

   (See note on Problem 5.)

10. If the freight train traveled z miles per hour, then the passenger train traveled $z + 20$ miles per hour, the freight train went $5z$ miles, the passenger train went $5(z + 20)$ miles, and $5(z + 20) = 5z + 100$.

11. If there are h half-pint cartons, then there are $6h$ pint cartons and

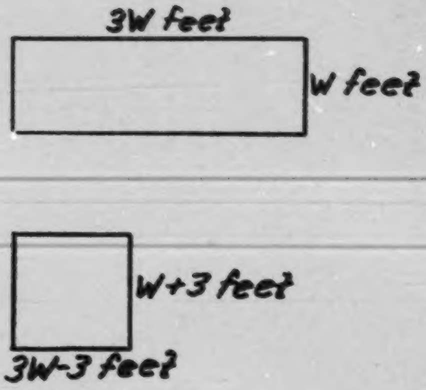
    $h\frac{1}{4} + (6h)\frac{1}{2} = 39$ (Sentence about quarts.)

    or $h \cdot \frac{1}{2} + (6h) = 78$ (Sentence about pints.)

    or $h + 2(6h) = 156$. (Sentence about half-pints.)

12. If y is the number of years until the men earn the same salary, then
    Mr. Brown will be earning $3600 + 300y$ dollars,
    Mr. White will be earning $4500 + 200y$ dollars, and
    $3600 + 300y = 4500 + 200y$.

13. If the table is w feet wide, then it is $3w$ feet long, the square would be $w + 3$ feet wide and $3w - 3$ feet long, and $w + 3 = 3w - 3$.



14. The following problems are, of course, only suggestions.

(a) I have $4.80 in coins of which there are two more dimes than nickels and twice as many half dollars as nickels. How many have I of each kind of coin?

(b) How wide is a rectangle which is three times as long as it is wide and which has an area of 300 square inches?

(c) A school ordered 85 booklets, some of which cost $.60 each and the rest cost $1.10 each. The total bill was $78.50. How many of each booklet were ordered?

(d) A strip of wood 69 inches long is to be cut into three pieces such that the second piece is 3 inches longer than the first and the third piece is 6 inches longer than the first. How long should the shortest piece be?

(e) Two boys caught 18 fish. Since one boy's family was twice as large as the other's, they decided to divide the 18 fish so that one boy took home twice as many fish as the other one did. How many fish did each boy take?

(f) A nut and bolt together weigh 5 ounces. If a pile of 30 nuts and 4 bolts weighs 59 ounces, how much does one bolt weigh?

(Notice that to make the open sentence true h cannot be a whole number. Thus a problem in which h is a number of people, for instance, would not be appropriate.)

(g) If Joe mowed $\frac{1}{4}$ of the lawn and Jerry mowed $\frac{1}{3}$ of the lawn, how much of the lawn must Jake mow in order to finish mowing the whole lawn?

15. If the smaller boat can carry n passengers, then the larger boat can carry $n + 80$ passengers, and $n + (n + 80) = 300$.

The Macy family drove to their cousin's. On the way back they took a side trip which added 80 miles to the distance. When they got home they had traveled a total of 300 miles. How far was it to their cousin's?

4-3. Open Sentences Involving Inequalities

Page 86. Here we extend the student's experience with translating to sentences involving inequalities. While we are still not trying to find the truth set of the open sentence, we may notice how the inequalities may have many numbers in the truth set instead of just one.

Answers to Problem Set 4-3a; page 87:

1. Teddy is more than 3 years old. How old is Teddy?
2. If Elizabeth were one year older, she still would be younger than 17 years old. How old is Elizabeth?
3. The quarters in my pocket are worth at most $1.75. How many quarters are in my pocket?
4. If we were to buy three more books at $5 each, the cost would be less than $100. How many books are we buying?

5. If the population of Metropolis increases by 10,000, the population will be more than 160,000. What is the present population of Metropolis?

6. Mr. Thompson has a box of thumb tacks, Mr. Ford has two boxes of them, and Miss Sneed has three boxes of them. Altogether they have at least 48 thumb tacks. How many thumb tacks are there in a box?

7. The guide walked the scenic trail once on Friday, twice on Saturday, and three times on Sunday. He walked a total of 50 miles. How long was the trail?

   (Note that the open sentence will not be true for a whole number. Note also in Problems 6 and 7 that a, $2a$, and $3a$ cannot represent sides of a triangle, since the sum of two of them is not greater than the third. If possible, try to lead your students to observe restrictions of this sort.)

*8. If three more boys join the club, the club will have more than 5 and less than 10 members. How many members has the club now?

9. If the whole group of boys is divided into two equal teams, each team is smaller than a full baseball team. How many boys were in the whole group?

*10. A 12-acre piece of land was divided between two sons in such a way that one son received more than three times as much as the other son. How many acres did each son receive?

Page 87. Example 2: A class of more capable students may enjoy a discussion of the fact that this problem is more completely described by the compound sentence

$$d > 0 \text{ and } (d - 32) + d \leq 48.$$

Since the phrase $d - 32$ is meaningless for $d < 32$, the domain is the set of numbers greater than or equal to 32.

Page 88. Example 3: There is, of course, a third condition:

$$5 < n + 6.$$

Since this sentence is true for all positive values of n, it adds nothing to the information about the sides of the triangle. We mention it here in case one of your students asks about it.

Some student may ask also about finding the truth set of

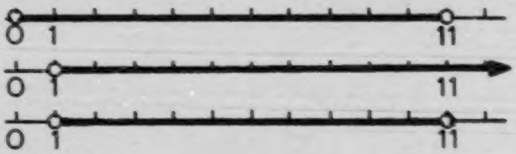
$$n < 5 + 6 \quad \text{and} \quad 6 < n + 5.$$

While we are not emphasizing the truth sets here, we could show the interested student how we can use graphs to find this truth set.

$n < 5 + 6$

$6 < n + 5$

$n < 5 + 6$ and $6 < n + 5$



Hence, the length of the third side is less than 11 inches and greater than one inch.

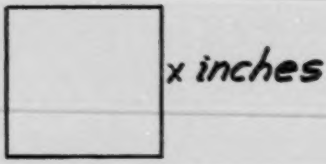
The main difficulty for the student in this might be seeing the graph of $6 < n + 5$. He may naturally observe, however, that this is equivalent to $n > 1$.

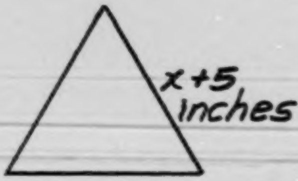
Answers to Problem Set 4-3b; pages 88-89:

1. If the number is n, then $\frac{3}{4}n + \frac{1}{3}n \geq 26$.

2. If Norman is y years old, then Bill is $y + 5$ years old, and $y + (y + 5) < 23$.

3. If a side of the square is x inches long, then a side of the triangle is $x + 5$ inches long, and $4x = 3(x + 5)$.





4. If the rate of the current is c miles per hour, then $c + 12$ is the rate of the boat downstream, and
$c + 12 < 30$.

5. If John can spend x hours on the job, then
$x > 2$ and $x \leq 4$.

6. If there are c minutes allowed for commercials, then
$c \geq 3$ and $c < 12$.
Time for material other than advertising is $30 - c$.
Another way of expressing the last idea is $T \leq 30 - 3$ and $T > 30 - 12$.

7. If there are t students in the class, then
$3t \geq t + 26$.

8. If Harry has d dollars, then Dick has $d + 15$ dollars, and Tom has $2(d + 15)$ dollars, and
$d + (d + 15) + 2(d + 15) = 205$.

9. If Harry has d dollars, then Dick has $d + 15$ dollars, and Tom has $2(d + 15)$ dollars, and
$d + (d + 15) + 2(d + 15) \leq 225$ and $d + (d + 15) + 2(d + 15) \geq 205$.

*10. If his score on the third test is t, then

$$\frac{75 + 82 + t}{3} \geq 88.$$

$$\frac{75 + 82 + 100}{3} = \frac{257}{3} = 85\frac{2}{3}$$

$$\frac{75 + 82 + 0}{3} = \frac{157}{3} = 52\frac{1}{3}$$

11. If there are s students in Scott School and m students in Morris School, then
(a) $s > m$
(b) $s = m + 500$

Page 90. This list of problems reviews the ideas of this chapter and indirectly reviews many of the ideas of the earlier chapters. The teacher must use judgment as to how many of these are done and when they are done. Some of them might be assigned one or two at a time along with later assignments as the class goes on into the next chapter. Look over the whole list to see what variety of problems you want your class to experience.

Answers to Review Problems; pages 90-96:

As opportunity presents itself, discuss with the students the restrictions on the domain of the variable which are imposed in some of the problems by the nature of the situation.

1. (a) If x is the number of cents in the collection before I contributed, then the phrase is: The total number of cents in the collection after I put in my fifteen cents. With this translation, the domain is the set of whole numbers.

   (b) If p is the number of dollars in the list price of a book, then the phrase is: The number of dollars paid for five copies of the same book, three bought at the list price and two bought at the reduced price of one dollar below list price.

   (c) If the boat has traveled t hours at ten miles per hour, then the phrase is: The number of miles remaining of a sixty-mile boat trip at the end of t hours.

   (d) If r is the number of 15-cent ice-cream cones, then the phrase is: The total number of cents spent by a group of 9 students at the recreation park, where each student bought r cones and each bought 25 cents' worth of popcorn.

   (e) If b is the number of inches in the length of the base of a triangle, then the phrase is: The number of square inches in the area of the triangle, where the altitude is 4 inches less than 3 times the length of the base.

2. (a) If n is the number, then the phrase is $n - 3$.
   (b) If Sam is y years old now, then the phrase is $y + 7$.
   (c) If Mary is x years old now, then the phrase is $x - 10$.
   (d) If t is the number of degrees of the present temperature, then the phrase is $t + 20$.
   (e) The phrase is $5n$.

3. (a) The phrase is $10x + 5y + 6$.
   (b) If the number is n, then the phrase is $n + 2n$.
   (c) If the first number is x and the second is y, then the phrase is $x + 2y$.
   (d) The phrase is $7w$.
   (e) If x is the population of a city in Kansas, then the phrase is $2x + 1,000,000$.
   (f) The phrase is $12x$.

4. (a) If b is the number of dollars in Betty's allowance, then the phrase is $2b + 1$.
   (b) The phrase is $40h$.
   (c) The phrase is $\frac{y}{1000}(25)$.
   (d) If Earl weighs e pounds, then the phrase is $e + 40$.
   (e) If the rectangle is n inches wide, then the phrase is $n(n + 3)$.

5. (a) The phrase is $1.59x$.
   (b) The phrase is $.75z$.
   (c) The phrase is $33.2g$.
   (d) The phrase is $29x + 59y$.
   (e) The phrase is $10t + u$.

6. (a) If x is the number of years of his age, then he is less than 80 years old.
   (b) If he earns y dollars in a year, his annual salary is 3600 dollars.
   (c) If z is the number of dollars of a bank's assets, then the assets are more than 100 million dollars.
   (d) If u, v, w are the numbers of degrees of the angles of a triangle, the sum of the angles is 180 degrees.
   (e) If the shorter side of a rectangle is z inches, and the longer side is 18 inches longer than this, then the area of the rectangle is 360 square inches.

7. (a) If Mary's sister is y years old, then

$$16 = y + 4.$$

   (b) If Bill bought b bananas, then

$$9b = 54.$$

   (c) If the number is n, then

$$2n + n < 39.$$

   (d) If Betty's allowance is b dollars, then Arthur's allowance is $2b + 1$ dollars, and

$$2b + 1 = 3b - 2.$$

   (e) $40t = 260$
   (f) $\frac{300}{t} \leq 50$

8. (a) If Pike's Peak is h feet high, then

$$h > 14{,}000.$$

   (b) $1.4 = 0.003n + 2(.1)$
   (c) If p is the number of people in a city in Colorado, then $2{,}000{,}000 > 2p$.
   (d) $x^2 > (x - 1)(x + 1)$
   (e) If y is the number of dollars in the valuation of the property, then $\frac{y}{1000}(24) = 348$.
   (f) If Earl weighs w pounds, then

$$w + 40 \leq 152.$$

9. (a) If the whole number is n, then its successor is $n + 1$, and $n + (n + 1) = 575$.
   (b) If the whole number is n, then its successor is $n + 1$, and $n + (n + 1) = 576$.
   This sentence is false for all whole numbers.
   If a number is odd, its successor is even; if the number is even, its successor is odd.
   Hence their sum cannot be even.
   (c) If the first number is n, then the second number is $n + 1$, and $n + (n + 1) = 576$.
   There is a number for which this sentence is true, since the domain of the variable is not restricted to whole numbers.
   (d) If one piece of the board is f feet long, then the other piece is $2f + 1$ feet long, and $f + (2f + 1) = 16$.
   (e) $3x = 225$

10. If the tens' digit is t, and the units' digit is u, then the number is $10t + u$, and $10t + u = 3(t + u) + 7$.

11. $42 - n$

12. (a) If n is the number, then $3(n + 17) = 192$.
    (b) If n is the number, then $3(n + 17) < 192$.

13. If the first number is x, then the second number is $5x$, and $x + 5x = 4x + 15$.

14. If Sally has K books, then Sue has $K + 16$ books, and $K + (K + 16) > 28$.

15. In one hour he can plow $\frac{1}{7}$ of the field with the first tractor. In two hours, using both tractors, he can plow $\frac{2}{7} + \frac{2}{5}$ of the field. The part of the field left unplowed is $1 - (\frac{2}{7} + \frac{2}{5})$. The open sentence is $\frac{x}{7} + \frac{x}{5} = 1$.

*16. If t is the number of the hour when you leave New York, then $t + h$ is the number of the hour, New York time, when you arrive in Los Angeles.
$t + h - 3$ is the number of the hour of arrival, Los Angeles time, and
$t + h - 3 < 12$.

17. Mr. Brown's weight m months ago was $175 + 5m$.
$175 + 5m = 200$.

18. (a) If n is a whole number, then $n + 1$ is its successor, and $n + (n + 1) = 45$.
(b) If n is an odd number, then $n + 2$ is the next consecutive odd number, and $n + (n + 2) = 75$. See whether your students notice that this can never be true for any odd number, since the sum of two odd numbers is even.

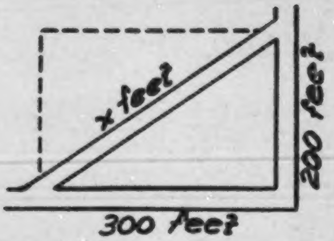
19. If the marked price was m dollars, then $176 = m - \frac{12}{100}m$.

20. If x is the number of dollars for one hour's work at the normal rate, then $\frac{3}{2}x$ is the number of dollars for one hour's work at the over-time rate, and
$40x + 8(\frac{3}{2}x) = 166.40$.

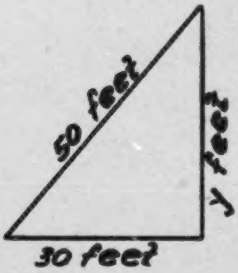
21. If the target is d feet away, then it took the bullet $\frac{d}{1700}$ seconds to reach the target, and it took the sound $\frac{d}{1100}$ seconds to come back, and $\frac{d}{1700} + \frac{d}{1100} = 2$.

An alternate solution which has the advantage of resulting in a simpler sentence but the disadvantage of the variable representing a number other than the one asked for is: If t is the time in seconds for the bullet to reach the target and $2 - t$ is the time in seconds for the sound to come back, then $1100(2 - t) = 1700t$.

22. If the short cut is x feet long, then $(200)^2 + (300)^2 = x^2$.



23. If y is the number of feet in the height of the pole, then, by the Pythagorean Theorem, $(30)^2 + y^2 = (50)^2$.



24. (a) $35 + 20t$
    (b) $h - 1$ is the number of one-hour periods after the initial hour, and the phrase is $35 + 20(h - 1)$.

25. If the radiator originally contains w quarts of water, it contains $w + 2$ quarts of mixture after the alcohol was added. Since 20% of this mixture is alcohol, there are $\frac{20}{100}(w + 2)$ quarts of alcohol in the mixture.

$$2 = \frac{20}{100}(w + 2)$$

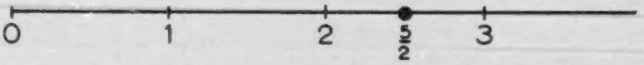
26. (a) $100x + 40y$
    (b) $100(2 \times 60) + 40y$
    (c) $100x + 40y = 20,000$

27. (a) $20 + 2w$
    (b) $12 + 3w$
    (c) $20 + 2w = 12 + 3w$

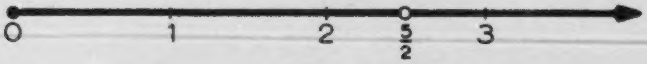
28. (a) $20 + \frac{h}{1500}$

(b) $20 + \frac{y}{1500} = 24$

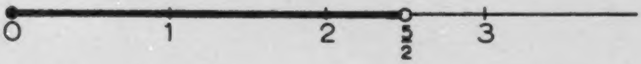
29. $x = \frac{5}{2}$



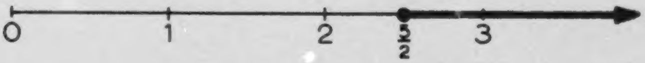
$x \neq \frac{5}{2}$



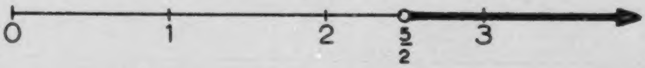
$x < \frac{5}{2}$



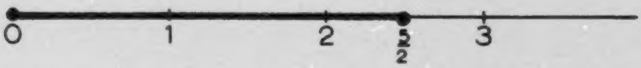
$x \nless \frac{5}{2}$



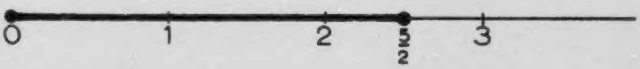
$x > \frac{5}{2}$



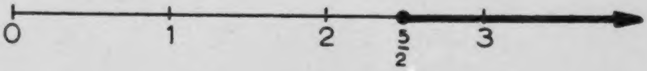
$x \ngtr \frac{5}{2}$



$x \leq \frac{5}{2}$



$x \geq \frac{5}{2}$

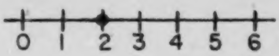


30. (a) $x > 2$

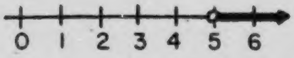
(b) $x \leq 6$

(c) $x > 2$ and $x \leq 6$

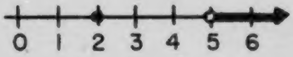
31. (a) $x = 2$



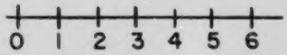
$x > 5$



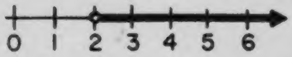
$x = 2$ or $x > 5$



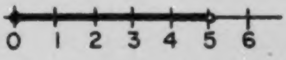
(b) $x = 2$ and $x > 5$



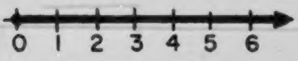
(c) $x > 2$



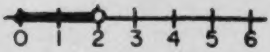
$x < 5$



$x > 2$ or $x < 5$



(d) $x < 2$ and $x < 5$



32. $(x \geq 1$ and $x < 3)$ or $(x > \frac{7}{2})$

33. This problem reviews sums of pairs of members of a set; it is not a problem set up primarily to get an answer. The pupil who tries to write an open sentence will find he is wasting his time. Instead he should observe that the man has a set of four members: {1.69, 1.95, 2.65, 3.15} and that he should examine the set of all possible sums of pairs of elements of the set.

+	1.69	1.95	2.65	3.15
1.69	3.38	3.64	4.34	4.84
1.95	3.64	3.90	4.60	5.10
2.65	4.34	4.60	5.30	5.80
3.15	4.84	5.10	5.80	6.30

This is the set: {3.38, 3.64, 3.90, 4.34, 4.60, 4.84,
5.10, 5.30, 5.80, 6.30}

From this we see: (a) The smallest amount of change he could have is 5.00 - 4.84, or 16 cents.

(b) The greatest amount of change possible is 5.00 - 3.38, or $1.62.

(c) There are four pairs of two boxes he cannot afford: one of $1.95 and one of $3.15; one of $2.65 and one of $3.15; two of $2.65; two of $3.15.

*34. If a is the units digit of one number then $10 + a$ is the number. If b is the units digit of the other number then $10 + b$ is the other number. If p is the product of the two numbers, then $p = \big((10 + a) + b\big)10 + ab$.

We can verify the above sentence by showing $p = (10 + a)(10 + b)$.

Thus, remembering that we can regard 10 + a as a single number,

$$\begin{aligned} p &= \big((10 + a) + b\big)10 + ab \\ &= (10 + a)10 + b(10) + ab && \text{distributive} \\ &= (10 + a)10 + 10b + ab && \text{commutative for multiplication} \\ &= (10 + a)10 + (10 + a)b && \text{distributive} \\ &= (10 + a)(10 + b) && \text{distributive} \end{aligned}$$

Chapter 4

Suggested Test Items

Write an open phrase or open sentence for each of the following:

1. The volume V in cubic feet of a rectangular solid whose dimensions are x yards, y feet, and z feet.
2. If a boy has 250 yards of chicken fence wire, how long and how wide can he make his chicken yard, if he would like to have the length 25 yards more than the width?
3. In an orchard containing 2800 trees, the number of trees in each row is 10 less than twice the number of rows. How many rows are there?
4. Paul has d dimes and three times as many quarters as dimes. Write an open phrase for the number of cents Paul has.
5. If h is the third of five consecutive whole numbers, then

    the fifth number is______,
    the second number is______,
    the sum of the first and the fifth number is______.

6. Jack is 3 years older than Ann, and the sum of their ages is less than 27 years. How old is Ann?
7. There are five large packages and three small ones. Each large package weighs 4 times as much as each small one; and the eight bundles together weigh 34 pounds 8 ounces. What is the weight of each bundle?

8. Jim has four times as much money as Walter. If he gives Walter 39 cents, the two boys will then have equal amounts. How much does Walter have now?

9. A father earns twice as much per hour as his son. If the father works for 8 hours and the son for 5 hours, they earn less than \$30. How much does the son earn per hour?

10. Separate \$38 into two parts such that one part is \$19 more than the other.

11. A basket full of peaches costs \$2.10. The peaches cost two dollars more than the basket. How much does the basket cost?

12. The thickness of a number of pages of a book if each page is $\frac{1}{400}$ of an inch thick.

13. The product of a whole number and its successor is 342. What is the number?

14. Bill's weight, 193 pounds, is at least 10 pounds more than Dave's. How much does Dave weigh?

15. A salesman insists that he must have at least \$12 a day in an expense account, and his boss insists that he must be given less than \$30 a day. Express this in an open sentence.

16. Complete the following problem so that the problem corresponds to the given open sentence.
Problem. John found a box containing \$2.40.

Open Sentence. $5(x + 4) + 10(4x) + 25x = 240$

17. Translate the following into open phrases or sentences using a single variable in each.
    (a) \$30 more than Tom's weekly salary
    (b) Tom's weekly salary is more than \$30.
    (c) Tom's weekly salary is \$30 more than Jim's. Together they earn \$140 per week.
    (d) Tom's weekly salary is \$30 more than Jim's. Together they earn more than \$140 per week.
    (e) Tom's weekly salary is \$30 more than Jim's. The sum of their weekly salaries is between \$140 and and \$200.

18. Write each of the open phrases which results from obeying the following directions:
    (a) Choose a number and add 4 to it to get a second number.
    (b) Subtract 3 from the second number to get a third number.
    (c) Find the average of the second and third numbers.

Chapter 5

THE REAL NUMBERS

5-1. The Real Number Line

In Chapters 1 to 4 the student has been discovering and applying properties of operations on a set of numbers which we called the numbers of arithmetic. This set consists of 0 and the numbers assigned to the points on the right of 0 on the number line. His work with familiar numbers gave him a secure hold on such concepts as the associative, commutative, and distributive (ACD) properties, open sentences, truth sets, etc. With this background he is now ready to give names to numbers which we assign to points on the left of 0. The total set of numbers corresponding to all points of the line, that is, the set of real numbers, is now his field of activity.

It should be mentioned that in this course we have chosen to approach the negative numbers in a manner different from some writers. Instead of presenting a new set of numbers (the real numbers) and then identifying a particular subset of these (the non-negative) with the original set (the numbers of arithmetic), we have chosen the following approach. We extend the numbers of arithmetic to the set of real numbers by attaching the negative numbers to the familiar numbers of arithmetic. This has several advantages: First, we do not need to distinguish between "signed" and "unsigned" numbers; to us the non-negative real numbers are the numbers of arithmetic. Second, it is not necessary for us to prove that the familiar properties hold for the non-negatives, for these properties are carried over intact along with the numbers of arithmetic. In this manner we avoid the confusion of establishing an "isomorphism" between positive numbers and "unsigned numbers". Notice that we have no need whatsoever for the ambiguous word "sign".

In general, we have taken the point of view that a ninth grade student really has some experience with negative numbers. He is quite ready to label the points to the left of 0 and, in so doing, make the extension to which we referred.

A general reference for the teacher for this chapter is Haag, *Studies in Mathematics*, *Volume III*, *Structure of Elementary Algebra*, Chapter 3, p. 3.16; pp. 3.23 - 3.26.

In this chapter we familiarize the student with the total set of real numbers. We include the order of real numbers, comparison of real numbers, and the operation of determining the *opposite* of a real number. Another section is devoted to a definition and discussion of the *absolute value* of a real number. Here it should be pointed out that the spiral technique is being applied. This first taste of absolute value is followed in each succeeding chapter by more and more uses and levels of abstraction of absolute value. The teacher does not need to go all out here on absolute value, because the topic reappears regularly.

Then in Chapters 6 and 7 we shall define addition and multiplication of real numbers, being careful to choose our definitions so that the familiar ACD properties of the operations will hold for all real numbers. In the remainder of the course the operations on real numbers will lead to many topics usually considered in elementary algebra courses.

Page 97. We introduce the negative numbers in much the same way that we labeled the points on the right side of the number line, which correspond to the numbers of arithmetic. Our notation for negative four, for example, is $^{-}4$, and we definitely intend that the dash "$^{-}$" be written in a raised position. At this stage, we do *not* want the student to think that something has been done to the number 4 to get the number $^{-}4$, but rather that $^{-}4$ is a *name* of the number which is assigned to the point 4 units to the left of 0 on the number line. In other words, the raised dash is *not* the symbol of an operation, but only an identifying *mark* for numbers to the left of zero.

In Section 5-3, the student *will* be able to think of $^{-}4$ as the number obtained from 4 by an operation called "taking the opposite". The opposite of 4 there introduced will be symbolized as -4, the dash being written in a centered position, and -4 will turn out to be a more convenient name for $^{-}4$.

[page 97]

Since each number of arithmetic has many names, so does each negative number. For example, the number $^-7$ has the names

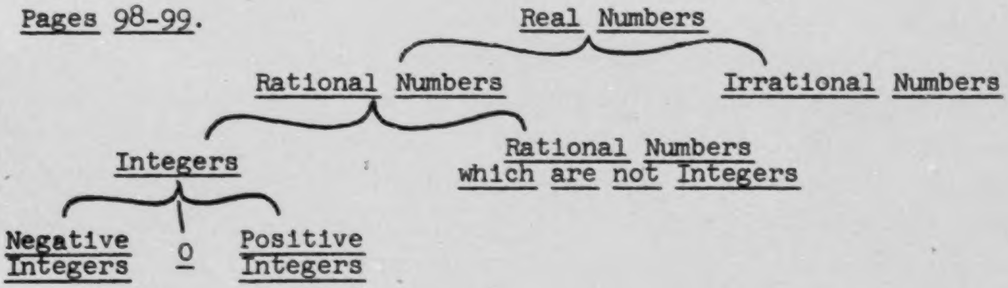
$$^-\frac{14}{2}, \quad ^-(7 \times 1), \quad ^-(\frac{3+2}{8} \times \frac{56}{5}), \text{ etc.}$$

In graphing real numbers, the student should be aware that the number line pictured which he draws is only an *approximation* to the true number line situation. Consequently, any information which he deduced from his number line picture is only as accurate as his drawing.

Page 98. Once the negative numbers have been introduced, we have the objects with which the student will be principally concerned throughout the next four years of his mathematics education. It is much too cumbersome to have to refer to "the numbers which correspond to all the points on the number line" or to "the numbers of arithmetic and their negatives", and so we use the customary name *the real numbers*. Do not let the students attach significance to the word "real". It is simply the name of this set of numbers.

Do not let the class discussion digress too far into irrational numbers. The proof that $\sqrt{2}$ is not rational will be given in Chapter 11. In the meanwhile we simply want the student to see an irrational such as $\sqrt{2}$ or π and to be told that there are many more.

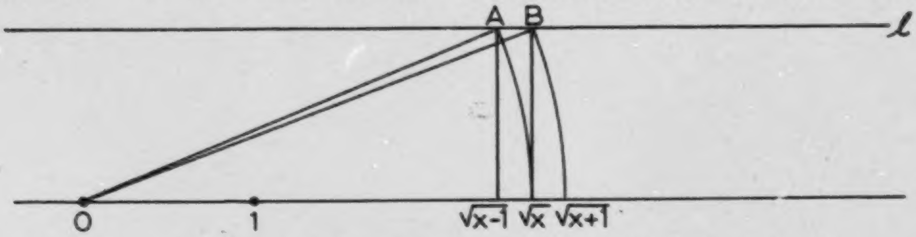
Pages 98-99.



A common misunderstanding is that some numbers on the line are real and others are irrational. The student should be encouraged to say, at least for the time being, that "$^-2$ is a real number which is a rational number and a negative integer; $\frac{3}{2}$ is a real number which is a rational number; $^-\sqrt{2}$ is a real number which is a negative irrational number". The point is that every integer is also a rational number and a real number; every rational number is a real number; and every irrational number is a real number.

Page 99. The method of locating $\sqrt{2}$ on the number line depends on the student's understanding of the Pythagorean Theorem. Again, this should not be allowed to distract from the main ideas of the section. If it would mean a complete lesson on the Pythagorean Theorem, it would be better to omit the construction.

For the more capable student the scheme given in the text for graphing $\sqrt{2}$ can easily be extended to give a method for determining successively $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, $\sqrt{6}$, etc. (The teacher need not feel compelled to pursue this.) Given the unit segment 0 to 1 on the number line, let ℓ be a line parallel to and one unit away from the number line. Suppose that x is a number greater than or equal to 1, for which we have been able to locate $\sqrt{x}$ on the number line. The perpendicular to the number line at $\sqrt{x}$ meets ℓ at a point B, and the circle with center O and radius OB then meets the arithmetic number line at $\sqrt{x+1}$. Applying the same technique to $\sqrt{x+1}$, we can locate $\sqrt{x+2}$. This process may be continued indefinitely. In particular, if $x = 1$, we can find successively $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, etc.



Similarly, the circle with center O and radius $\sqrt{x}$ meets ℓ at a point A, and the perpendicular to ℓ at A then meets the arithmetic number line at $\sqrt{x-1}$. If $x \geq 2$, the process may be repeated to get $\sqrt{x-2}$, (and so forth).

To illustrate, suppose $x = 16$. Since we can easily locate $\sqrt{16} = 4$ on the number line, the scheme above permits us to locate

$\sqrt{15}$ and $\sqrt{17}$. Applying the method to each of these, we can locate $\sqrt{14}$ and $\sqrt{18}$, (and so forth).

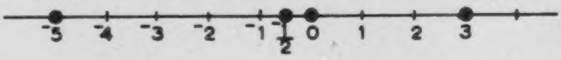
Page 99. The student may have acquired somewhere along the line a feeling that $\sqrt{2}$ is a rational number, and we hope to dispel this notion by having him compute the squares of several rational numbers which he has come to associate with $\sqrt{2}$. We hope that he comes to realize that $\sqrt{2}$ is a number between 1 and 2, between 1.4 and 1.5, between 1.41 and 1.42, between 1.414 and 1.415, and so on. We will prove in Chapter 11 that $\sqrt{2}$ is not a rational number.

Page 100. Is $\frac{1}{2}\sqrt{2}$ a rational number? $3 + \sqrt{2}$? In the case of $\frac{1}{2}\sqrt{2}$, help the student to reason as follows: If $\frac{1}{2}\sqrt{2}$ were a rational number, then $2 \times \frac{1}{2}\sqrt{2}$ would be a rational number; in other words, $\sqrt{2}$ would be a rational number. But $\sqrt{2}$ is not a rational number. Therefore, $\frac{1}{2}\sqrt{2}$ cannot be a rational number. Similarly, he should ascertain that $3 + \sqrt{2}$ is not a rational number, since if it were, then $\sqrt{2} = (3 + \sqrt{2}) - 3$ would be a rational number. Do not dwell on this point, but here we broach for the first time the idea of indirect proof (or proof by contradiction). A more detailed discussion will follow in Section 7-8.

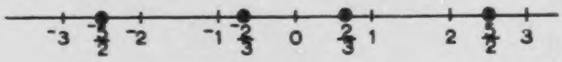
We want the student to realize that there are many points on the number line which do not have rational coordinates. He will eventually learn how to name some of these, but he should not worry about this at present. For a discussion of the representation of rational and irrational numbers as decimals, see Haag, Studies in Mathematics, Vol III, SMSG, Chapter 4 and Appendix A.

Answers to Problem Set 5-1; page 100:

1. (a) $\{0, 3, {}^{-}5, {}^{-}(\frac{1}{2})\}$

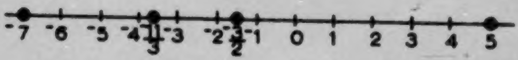


   (b) $\{{}^{-}(\frac{2}{3}), \frac{2}{3}, \frac{5}{2}, {}^{-}(\frac{5}{2})\}$

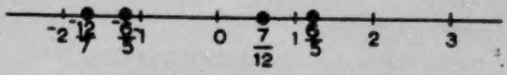


The points located will be only approximations, of course. The students should have the feeling, however, that there are points on the line corresponding to the numbers used in the exercises and that the number line pictures they draw are reasonable approximations to the graphs of the numbers.

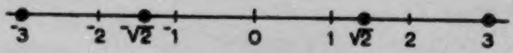
(c) $\{^-(\frac{3}{2}),\ 5,\ ^-7,\ ^-(\frac{11}{3})\}$


$^-7$
$^-6$
$^-5$
$^-4$
$^-\frac{11}{3}$
$^-3$
$^-2$
$^-\frac{3}{2}$
$^-1$
0
1
2
3
4
5

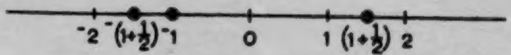
(d) $\{^-(\frac{12}{7}),\ ^-(\frac{6}{5}),\ \frac{6}{5},\ \frac{7}{12}\}$


$^-2$
$^-\frac{12}{7}$
$^-\frac{6}{5}$
$^-1$
0
$\frac{7}{12}$
1
$\frac{6}{5}$
2
3

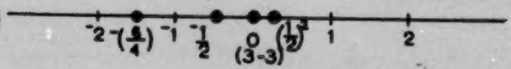
(e) $\{\sqrt{2},\ ^-\sqrt{2},\ 3,\ ^-3\}$


$^-3$
$^-2$
$^-\sqrt{2}$
$^-1$
0
1
$\sqrt{2}$
2
3

(f) $\{^-1,\ ^-(1 + \frac{1}{2}),\ (1 + \frac{1}{2})\}$


$^-2$
$^-(1+\frac{1}{2})$
$^-1$
0
1
$(1+\frac{1}{2})$
2

(g) $\{^-(\frac{1}{2}),\ (\frac{1}{2})^2,\ ^-(\frac{6}{4}),\ (3 - 3)\}$


$^-2$
$^-(\frac{6}{4})$
$^-1$
$^-\frac{1}{2}$
0
(3-3)
$(\frac{1}{2})^2$
1
2

2. (a), (b), (d), (e), (g), (h), and (j) follow from the fact that every positive number is to the right of 0 and every negative number is to the left of 0 on the real number line.

(a) 3 is to the right of $^-4$.

(b) 5 is to the right of $^-4$.

(c) $^-2$ is to the right of $^-4$.

(d) 1 is to the right of $^-\sqrt{2}$.

(e) 0 is to the right of $^-(\frac{5}{2})$.

(f) $^-(\frac{5}{2})$ and $^-(\frac{10}{4})$ are names for the same number and, so, name the same point on the number line.

(g) 3 is to the right of 0.

(h) $\sqrt{2}$ is to the right of $^-4$.

(i) $^-(\frac{21}{4})$ is to the right of $^-(\frac{16}{3})$.

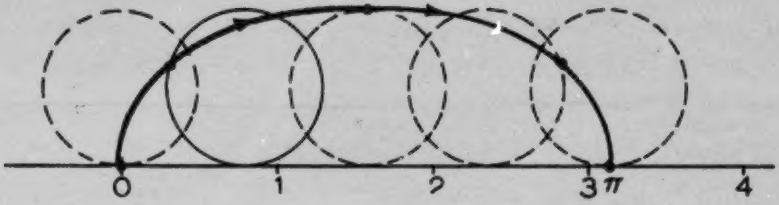
$^-(\frac{21}{4} \times \frac{3}{3}) = {}^-(\frac{63}{12})$ and $^-(\frac{16}{3} \times \frac{4}{4}) = {}^-(\frac{64}{12})$.

$^-(\frac{63}{12})$ is to the right of $^-(\frac{64}{12})$.

Because of the manner in which the negative numbers were constructed on the number line, the negative number, $^-(\frac{63}{12})$, corresponding to the lesser of the two numbers of arithmetic, is to the right of $^-(\frac{64}{12})$.

(j) $\frac{1}{2}$ is to the right of $^-(\frac{1}{2})$.

3.



Since "rolling the circle" gives us a length on the number line equal to the circumference of the circle, the circle comes to rest at a point on the line π units to the right of 0. If the circle is rolled to the left one revolution the circle will come to rest at a point on the number line whose coordinate is $^{-}\pi$.

4. (a) $^{-}2$ is an integer, a rational number, a real number.

   (b) $^{-}(\frac{10}{3})$ is a rational number and a real number.

   (c) $-\sqrt{2}$ is a real number.

5. A = {0, 1, 2, 3, . . .}
   B = {1, 2, 3, 4, . . .}
   C = {0, 1, 2, 3, . . .}
   I = {. . ., -3, -2, -1, 0, 1, 2, 3, . . .}
   N = {1, 2, 3, . . .}

   A, the set of whole numbers, and C, the set of non-negative integers, are the same since both are comprised of the positive integers and 0. B, the set of positive integers, and N, the set of counting numbers, are the same, since both consist of the counting numbers 1, 2, 3, ...

5-2. Order On The Real Number Line

We believe that the student will expect the relation "is less than" for the real numbers to have the same meaning as it did for what we now call the non-negative real numbers.

It is possible that some students may wonder if there is some other definition of "is less than" which might have been chosen. If they raise the question, the following discussion of an interpretation which at first glance seems plausible may be helpful. Although "is less than" was defined to mean "is to the left of" on the number line for the numbers of arithmetic, it could clearly be there interpreted as "is closer to 0 than". It is then plausible that "<" for the real numbers might well have this latter meaning! On the other hand, the example of the thermometer does not agree with this interpretation, nor would such familiar things as the variation in the height of tides or elevations above and below sea level.

There is also a good mathematical reason for rejecting this interpretation. The mathematician is never really interested in a relation as such, but rather in the properties it enjoys. Whatever meaning is attached to "is less than" we want to be able to say, for example, that precisely one of the sentences "$3 < {}^-3$" and "${}^-3 < 3$" is true. The plausible interpretation does not permit this comparison, since neither of ${}^-3$ and 3 is closer to 0 than the other.

Page 102. We expect the student to say that

$<$ means "is to the left of",
$\leq$ means "is to the left of or is the same number as",
$\geq$ means "is to the right of or is the same number as",
$\nless$ means "is not to the left of",
$\ngtr$ means "is not to the right of",

on the real number line. He should definitely have a feeling for the meaning of "<" since much of the further discussion of order is framed for "is less than".

In attempting to compare negative rational numbers, the students should be aware that the multiplication property of 1 can be applied to convert these into rational numbers represented by fractions with the same denominator. Thus, to compare ${}^-(\frac{7}{13})$ and ${}^-(\frac{9}{17})$, we have

$$
\begin{aligned}
{}^-\left(\frac{7}{13}\right) &= {}^-\left(\frac{7}{13} \times \frac{17}{17}\right) = {}^-\left(\frac{119}{13 \times 17}\right), \\
{}^-\left(\frac{9}{17}\right) &= {}^-\left(\frac{9}{17} \times \frac{13}{13}\right) = {}^-\left(\frac{117}{17 \times 13}\right).
\end{aligned}
$$

Since $13 \times 17 = 17 \times 13$ by the commutative property of multiplication, it is easy to compare the numbers

$$^-\left(\frac{119}{13 \times 17}\right) \text{ and } ^-\left(\frac{117}{17 \times 13}\right)$$

Answers to Problem Set 5-2a; pages 102-104:

1. (a) $3 \leq {}^-1$ is false, since 3 is to the right of $^-1$ on the number line and is therefore greater than $^-1$. Note again the easy comparison of a positive number and a negative number.

   (b) $2 < {}^-(\frac{7}{2})$ is false, since 2 is to the right of $^-(\frac{7}{2})$ and is therefore greater than $^-(\frac{7}{2})$.

   (c) $^-4 \nless 3.5$ is false.

   (d) $^-(\frac{12}{5}) < {}^-2.2$ is true. Changing the decimal fraction to a common fraction, the statement becomes $^-(\frac{12}{5}) < {}^-(\frac{22}{10})$. Now $^-(\frac{12}{5}) = {}^-(\frac{24}{10})$ and, so, is to the left of $^-(\frac{22}{10})$ on the number line. Thus $^-(\frac{12}{5})$ is to the left of $^-2.2$.

   (e) $^-(\frac{3}{5}) \leq {}^-(\frac{3+0}{5})$ is true, since $^-(\frac{3}{5})$ and $^-(\frac{3+0}{5})$ are names for the same number, they correspond to the same point on the number line. But any real number is less than or equal to itself!

   The truth status of sentences (f) - (j) can be determined in the same manner as (a) and (b) above; consequently, we simply list the results.

   (f) $^-4 \ngtr 3.5$ is true.

   (g) $^-6 > {}^-3$ is false.

   (h) $3.5 < {}^-4$ is false.

   (i) $^-3 < {}^-2.8$ is true.

   (j) $^-\pi \nless {}^-2.8$ is false.

2. The purpose of this exercise is to lead the student to compare two numbers. By the time the student completes the exercise, he should see that there is only one true sentence in each group of three sentences below.

(a) $^{-}3.14 < {}^{-}3$ is true.

$^{-}3.14 = {}^{-}3$ is false.

$^{-}3.14 > {}^{-}3$ is false.

$^{-}3.14$ is to the left of $^{-}3$ on the number line and is therefore less than $^{-}3$.

(b) $2 < {}^{-}2$ is false.

$2 = {}^{-}2$ is false.

$2 > {}^{-}2$ is true.

(c) Simplifying the names of the numbers $\frac{5 + 3}{2}$ and 2×2, we have the pair 4 and 4.

$4 < 4$ is false, or $\frac{5 + 3}{2} < 2 \times 2$ is false.

$4 = 4$ is true, or $\frac{5 + 3}{2} = 2 \times 2$ is true.

$4 > 4$ is false, or $\frac{5 + 3}{2} > 2 \times 2$ is false.

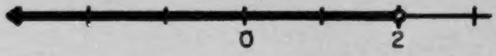
(d) If we write $^{-}0.001$ and $^{-}(\frac{1}{1000})$ as common fractions, we have the same number twice in the pair, instead of two distinct numbers. Thus,

$^{-}(\frac{1}{1000}) < {}^{-}(\frac{1}{1000})$ is false.

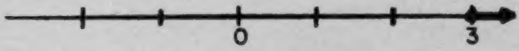
$^{-}(\frac{1}{1000}) = {}^{-}(\frac{1}{1000})$ is true.

$^{-}(\frac{1}{1000}) > {}^{-}(\frac{1}{1000})$ is false.

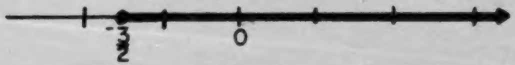
3. (a) $y < 2$.



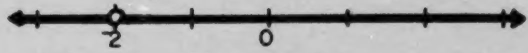
(b) $u \nless 3$.



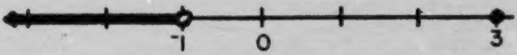
(c) $v \geq {}^{-}(\frac{3}{2})$



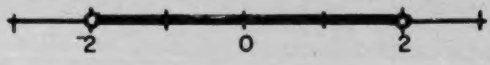
(d) $r \neq {}^{-}2$.



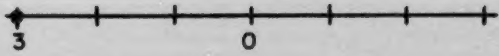
(e) $x = 3$ or $x < {}^{-}1$.



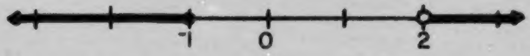
(f) $c < 2$ and $c > {}^{-}2$.



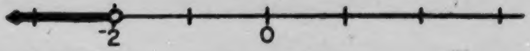
(g) $a \leq {}^{-}3$ and $a \geq {}^{-}3$.



(h) $d \leq {}^{-}1$ or $d > 2$.



(i) $a < 6$ and $a < {}^{-}2$.



(j) $u > 2$ and $u < {}^-3$. The truth set in this case is the empty set, $\emptyset$, and hence has no graph. The student should observe that there is no number both less than ${}^-3$ and greater than 2. The exercise may serve to remind the student to notice carefully whether the connective in the sentence is "and" or "or", and to interpret the sentence accordingly.

4. (a) A sentence whose truth set is the set of all real numbers not equal to 3 is: $y \neq 3$. Another is: $y > 3$ or $y < 3$. Note that although the sentences above are mathematically equivalent, their English translations differ, and it is the former sentence which describes the required set directly.

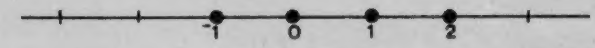
(b) A sentence whose truth set is the set of all real numbers less than or equal to ${}^-2$ is $v \leq {}^-2$. Another is: $v \ngtr {}^-2$.

(c) A sentence whose truth set is the set of all real numbers not less than ${}^-(\frac{5}{2})$ is $x \nless {}^-(\frac{5}{2})$. Another is $x \geq {}^-(\frac{5}{2})$. Notice here that the alternate form is easier to comprehend. This may suggest to the student a clearer description in English for the required set.

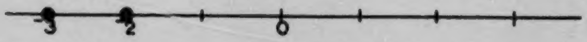
5. If p is any positive real number, and n is any negative real number, then n is to the left of zero, and p is to the right of zero; thus n is to the left of p. (Recall that this principle was used to speed the comparison of numbers in many of the preceding exercises.) It follows that "$n < p$" and "$n \neq p$" are true statements and "$p < n$" is false. The statement "$n \leq p$" is true since the statement means $n < p$ or $n = p$, and, though the second statement is false, the first is true.

6. If p is any number of the set of integers, the truth set of

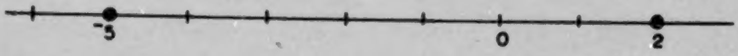
(a) ${}^-2 < p$ and $p < 3$ is $\{{}^-1, 0, 1, 2\}$. In words, this is the set of all integers both greater than ${}^-2$ and less than 3. On the number line the set would have the graph:



(b) $p \leq {}^-2$ and ${}^-4 < p$, is $\{{}^-2, {}^-3\}$. This is the set of all integers both less than ${}^-4$ and greater than or equal to ${}^-2$. Its graph is:



(c) $p = 2$ or $p = {}^-5$ is $\{{}^-5, 2\}$. In words we have the direct description of this set: the set of integers equal to 2 or ${}^-5$. Its graph is:



7. (a) A temperature rise of ten degrees from ${}^-5$ means that the temperature has risen from ${}^-5$ to 0, an increase of five degrees, and then has risen five more degrees, making the final temperature reading 5 degrees above 0.

(b) In rising five degrees from an initial reading of ${}^-10$ degrees, the temperature moves five degrees toward zero, making the final reading ${}^-5$ degrees.

(c) An increase in temperature from ${}^-15^\circ$ to 35° above zero involves a rise from ${}^-15$ to 0, that is, an increase of fifteen degrees and then a further rise from 0 to 35 above zero, an increase of thirty-five degrees. The total increase is then fifty degrees.

8. Use =, <, or > to relate each of the following pairs so as to make a true sentence.

(a) $\frac{3}{5} > \frac{{}^-6}{10}$.

Notice that in this and in most of the following exercises the multiplication property of 1 may be used to facilitate the comparison.

(b) $\frac{3}{5} > \frac{3}{6}$.

(c) $\frac{9}{12} > \frac{8}{12}$.

(d) $\frac{{}^-173}{29} > \frac{{}^-183}{29}$.

(e) $\frac{{}^-3}{5} < \frac{3}{6}$.

(f) $\frac{{}^-3}{5} < \frac{{}^-3}{6}$.

Page 105. In stating the *comparison property* and, later on, the *transitive property*, we are following the convention established in Chapter 2 to the effect that whenever we write a sentence about numbers, we are tacitly asserting the truth of that sentence.

The comparison property here given is also called the *trichotomy property* of order. Notice that it is a property of $<$; that is, given any two different numbers, they can be ordered so that one is less than the other. When the property is stated we must include the third possibility that the numerals name the same number. Hence, the name "trichotomy".

Although "$a < b$" and "$b > a$" involve different orders, these sentences say exactly the same thing about the numbers a and b. Thus, we can state a trichotomy property of order involving "$>$" as:

> For any number a and any number b,
> exactly one of these is true:
> $a > b$, $a = b$, $b > a$.

If, instead of concentrating attention on the order relation, we concentrate on the two numbers, then either "$a < b$" or "$a > b$" is true, but not both. Here we fix the numbers a and b and then make a decision as to which order relation applies. It is purely a matter of which we are interested in: the numbers or the order. The comparison property is concerned with an *order*.

Answers to Problem Set 5-2b; pages 105-106:

1. (a) $^-2 < {}^-1.6$.

   (b) $^-2 < 0$.

   At this point pupils should recall that if a number is negative it will be located to the left of 0 on the real number line; if positive, the right of 0. Consequently a negative number is always less than a positive number or zero. Using this fact makes easier Problem 1(c) and others in this problem set.

   (c) $^-\left(\frac{2 \times 3 \times 4}{5}\right) < \frac{2 \times 3 \times 4}{5}$.

   (d) $^-16 = {}^-\left(\frac{32}{2}\right)$.

(e) $12 = (5 + 2)(\frac{1}{7} \times \frac{36}{3})$

Note the use of the associative property in simplifying the second number:

$(5 + 2)(\frac{1}{7} \times \frac{36}{3}) =$

$7(\frac{1}{7} \times 12) =$

$7(\frac{1}{7})(12) =$

$12.$

(f) $^{-}2 < 2.$

2. Use only "<" to make true sentences.

(a) $^{-}3 < 2.$

(b) $\frac{^{-}5}{2} < \frac{4}{2}.$

(c) $\frac{^{-}6}{5} < \frac{^{-}4}{5}.$

(d) $\frac{4}{5} < \frac{11}{10}.$

(e) $\frac{^{-}4}{50} < \frac{11}{100}.$

(f) $\frac{205}{26} < \frac{103}{13}.$ Note that $\frac{103}{13} = \frac{206}{26}.$

(g) $\frac{2}{3} < \frac{13}{15}.$ $\frac{2}{3} = \frac{10}{15}$ and $\frac{10}{15} < \frac{13}{15}.$

(h) $^{-}(\frac{25}{238}) < \frac{12}{119}$

(i) $^{-}1.5 < {}^{-}\sqrt{2}$. Recall that $1.41 < \sqrt{2} < 1.42.$

*(j) $1.5 + 3 < \sqrt{2} + \pi.$

Here the student may write $1.5 + 3 = 1.4 + 3.1$ and use the inequalities $1.4 < \sqrt{2}$ and $3.1 < \pi$ to show that $1.4 + 3.1 < \sqrt{2} + \pi$. Finally, by the transitive property, $1.5 + 3 < \sqrt{2} + \pi$. Here he has anticipated the property "If a, b, c, and d are real numbers for which $a < b$ and $c < d$, then $a + c < b + d$", from Chapter 8.

3. If a is a real number and b is a real number, then exactly one of the following is true:

$$a = b, \ a > b, \ b > a.$$

Restating Exercise 1, using pairs of numbers given:

(a) In 1(a) either $^-2 > {}^-1.6$ or $^-1.6 > {}^-2$. Since $^-1.6$ is to the right of $^-2$, $^-1.6 > {}^-2$ is the true sentence.

(b) $0 > {}^-2$.

(c) $(\frac{2 \times 3 \times 5}{5}) > {}^-(\frac{2 \times 3 \times 4}{5})$.

(d) $^-16 = {}^-(\frac{32}{2})$.

(e) $12 = (5 + 2)(\frac{1}{7} \times \frac{36}{3})$.

(f) $2 > {}^-2$.

4. The best statement would be: "If a is a real number and b is a real number, then exactly one of the following is true: $a \geq b$ or $a < b$.

Some may say "For any real numbers a and b, $a \geq b$ or $a \leq b$. If $a \leq b$ and $b \leq a$, then $a = b$". The last sentence of this particular statement is reasonable and innocent in appearance. Surprisingly enough, it turns out to be one of the most useful criteria for determining that two variables have the same value! In many instances in the calculus, for example, one is able to show by one argument that $a \leq b$ and by another that $b \leq a$. He is then able to conclude that $a = b$. Given two numerals, it is usually trivial to check whether or not they name the same number. In the case of two numbers, of course, we have complete information. It is only when our information about two "numerals" is incomplete that a statement like, "If $a \leq b$ and $b \leq a$, then $a = b$", can possibly be useful as a tool.

Page 106. Any attempt to illustrate this transitive property of $<$ with triples of integers is likely to be met with a vociferous "So what!" by your students. On the other hand, not only can this property be illustrated with fractions as in the text, but the student can also begin to appreciate its usefulness and perhaps be inclined to say his "So what!" in a quieter voice.

The student will in the course of his mathematical training see many other relations which have a transitive property:

"is equal to" for numbers, "is a factor of" for positive integers, "is congruent to" for various geometric figures, etc.

What is an easy way to tell $\frac{337}{113} < 3$? By the multiplication property of 1, $3 = 3 \times \frac{113}{113} = \frac{339}{113}$, so that $\frac{337}{113} < 3$. Similarly, $3 = 3 \times \frac{55}{55} = \frac{165}{55}$, so that $3 < \frac{167}{55}$.

Answers to Problem Set 5-2c; Pages 106-107:

1. (a) ${}^{-}(\frac{1}{5}) < \frac{3}{2}$, $\frac{3}{2} < 12$, ${}^{-}(\frac{1}{5}) < 12$.

   (b) ${}^{-}\pi < {}^{-}\sqrt{2}$, ${}^{-}\sqrt{2} < \pi$, ${}^{-}\pi < \pi$.

   (c) ${}^{-}1.7 < 0$, $0 < 1.7$, ${}^{-}1.7 < 1.7$.

   (d) ${}^{-}(\frac{27}{15}) < {}^{-}(\frac{3}{15})$, ${}^{-}(\frac{3}{15}) < {}^{-}(\frac{2}{15})$, ${}^{-}(\frac{27}{15}) < {}^{-}(\frac{2}{15})$.

   (e)
$$\frac{12(\frac{1}{2} + \frac{1}{3})}{3} = \frac{12 \cdot \frac{1}{2} + 12 \cdot \frac{1}{3}}{3}$$
$$= \frac{6 + 4}{3}$$
$$= \frac{10}{3}$$

   Then the order is

$${}^{-}(\frac{5}{3}) < \frac{6}{3},\quad \frac{6}{3} < \frac{12(\frac{1}{2} + \frac{1}{3})}{3},\quad {}^{-}(\frac{5}{3}) < \frac{12(\frac{1}{2} + \frac{1}{3})}{3}$$

   (f)
$$\frac{3 \times (27 + 6)}{9} = \frac{3 \times 33}{9}$$
$$= 11 \text{ or } \frac{22}{2}.$$

$$\frac{(2 \times 3) + (7 \times 9)}{6} = \frac{6 + 63}{6}$$
$$= \frac{69}{6}$$
$$= \frac{23}{2}.$$

$$\frac{(99 \times 3)\frac{1}{3}}{2} = \frac{99 \times (3 \times \frac{1}{3})}{2}$$
$$= \frac{99}{2}.$$

Since $\frac{22}{2} < \frac{23}{2}$, $\frac{23}{2} < \frac{99}{2}$, and $\frac{22}{2} < \frac{99}{2}$, we find that

$$\frac{3 \times (27 + 6)}{9} < \frac{(2 \times 3) + (7 \times 9)}{6}$$

$$\frac{(2 \times 3) + (7 \times 9)}{6} < \frac{(99 \times 3)\frac{1}{3}}{2}$$

$$\frac{3 \times (27 + 6)}{9} < \frac{(99 \times 3)\frac{1}{3}}{2}$$

(g) $3^2 = 9$, $4^2 = 16$, and $(3 + 4)^2 = 49$. Therefore

$3^2 < 4^2$, $4^2 < (3 + 4)^2$, $3^2 < (3 + 4)^2$.

(h) $^-(\frac{1}{2}) = {}^-(\frac{6}{12})$, $^-(\frac{1}{3}) = {}^-(\frac{4}{12})$, $^-(\frac{1}{4}) = {}^-(\frac{3}{12})$. Thus

$^-(\frac{6}{12}) < {}^-(\frac{4}{12})$, $^-(\frac{4}{12}) < {}^-(\frac{3}{12})$, $^-(\frac{6}{12}) < {}^-(\frac{3}{12})$, and

$^-(\frac{1}{2}) < {}^-(\frac{1}{3})$, $^-(\frac{1}{3}) < {}^-(\frac{1}{4})$, $^-(\frac{1}{2}) < {}^-(\frac{1}{4})$

(i) $1 + \frac{1}{2} = \frac{3}{2}$ or $\frac{6}{4}$.

$1 + (\frac{1}{2})^2 = 1 + \frac{1}{4} = \frac{5}{4}$.

$(1 + \frac{1}{2})^2 = (\frac{3}{2})^2 = \frac{9}{4}$, or

$\frac{5}{4} < \frac{6}{4}$, $\frac{6}{4} < \frac{9}{4}$, $\frac{5}{4} < \frac{9}{4}$, or

$1 + (\frac{1}{2})^2 < 1 + \frac{1}{2}$, $1 + \frac{1}{2} < (1 + \frac{1}{2})^2$, $1 + (\frac{1}{2})^2 < (1 + \frac{1}{2})^2$.

2. Of three real numbers a, b, and c, if $a > b$ and $b > c$ then $a > c$.

Illustrations from Problem 1:

1(a) $^-(\frac{1}{5})$, $\frac{3}{2}$, 12: $12 > \frac{3}{2}$, $\frac{3}{2} > {}^-(\frac{1}{5})$, $12 > {}^-(\frac{1}{5})$

1(g) 3^2, 4^2, $(3 + 4)^2$: $(3 + 4)^2 > 4^2$, $4^2 > 3^2$, $(3 + 4)^2 > 3^2$.

3. Art is heavier than Bob.

Bob is heavier than Cal.

Conclusion: Art is heavier than Cal. Let Art's, Bob's and Cal's weights be represented respectively by the numbers

[pages 106-107]

a, b, c.
From the first sentence: $a > b$.
From the second sentence: $b > c$.
From the transitive property as stated in Problem 2, $a > c$, that is, Art is heavier than Cal.

4. The transitive property for "=" is: For all real numbers a, b, and c, if $a = b$ and $b = c$, then $a = c$.
If Art weighs the same as Bob and Bob and Cal are equal in weight, we know Art and Cal must weigh the same. If $3 + 4 = 7$ and $7 = 5 + 2$, then $3 + 4 = 5 + 2$.

5. The transitive property for $\geq$ would be: For all real numbers a, b and c, if $a \geq b$ and $b \geq c$, then $a \geq c$. If $\pi \geq 3.14$ and $3.14 \geq 2$, then $\pi \geq 2$.

6. (a) The non-positive real numbers are the set of numbers less than or equal to 0; in other words, the set comprised of 0 and all negative numbers.

(b) The non-negative real numbers are the set of numbers greater than or equal to 0; in other words, the set consisting of zero and all the positive numbers.

7. (a) $^{-}(\frac{15}{8})$ and $^{-}(\frac{25}{12})$. Some students may observe that $^{-}(\frac{15}{8}) = {}^{-}(1\frac{7}{8})$ and $^{-}(\frac{25}{12}) = {}^{-}(2\frac{1}{12})$. Referring to the number line, they will see that $^{-}(\frac{25}{12}) < {}^{-}(\frac{15}{8})$.

Some may reason as follows:

$^{-}(\frac{15}{8}) > {}^{-}(\frac{16}{8})$ or $^{-}2$.

$^{-}2$ or $^{-}(\frac{24}{12}) > {}^{-}(\frac{25}{12})$.

If $^{-}(\frac{15}{8}) > {}^{-}2$ and $^{-}2 > {}^{-}(\frac{25}{12})$, then $^{-}(\frac{15}{8}) > {}^{-}(\frac{25}{12})$.

(b) $^{-}(\frac{17}{35})$ and $^{-}(\frac{7}{13})$.

Note that $^{-}(\frac{17}{35}) > {}^{-}(\frac{1}{2})$ or $^{-}(\frac{17}{34})$, and $^{-}(\frac{1}{2})$ or $^{-}(\frac{7}{14}) > {}^{-}(\frac{7}{13})$.

If $^{-}(\frac{17}{35}) > {}^{-}(\frac{1}{2})$ and $^{-}(\frac{1}{2}) > {}^{-}(\frac{7}{13})$, then $^{-}(\frac{17}{35}) > {}^{-}(\frac{7}{13})$.

(c) ${}^{-}(\frac{145}{28})$ and ${}^{-}(\frac{104}{21})$.

Using mixed numbers, ${}^{-}(\frac{145}{28}) = {}^{-}(5\frac{5}{28})$ and ${}^{-}(\frac{104}{21}) = {}^{-}(4\frac{20}{21})$

If ${}^{-}(5\frac{5}{28}) < {}^{-}5$ and ${}^{-}5 < {}^{-}(4\frac{20}{21})$,

then ${}^{-}(5\frac{5}{28}) < {}^{-}(4\frac{20}{21})$ and ${}^{-}(\frac{145}{28}) < {}^{-}(\frac{104}{21})$

Alternatively,

If ${}^{-}(\frac{145}{28}) < {}^{-}(\frac{140}{28})$ or ${}^{-}5$, and ${}^{-}(\frac{105}{21})$ or ${}^{-}5 < {}^{-}(\frac{104}{21})$,

then ${}^{-}(\frac{145}{28}) < {}^{-}(\frac{104}{21})$.

(d) ${}^{-}(\frac{192}{46})$ and ${}^{-}(\frac{173}{44})$.

If ${}^{-}(\frac{192}{46}) < {}^{-}(\frac{184}{46})$ or ${}^{-}4$, and ${}^{-}(\frac{176}{44})$ or ${}^{-}4 < {}^{-}(\frac{173}{44})$,

then ${}^{-}(\frac{192}{46}) < {}^{-}(\frac{173}{44})$.

5-3. Opposites

Your students have quite likely observed by now that, except for 0, the real numbers occur as pairs, the two numbers of each pair being equidistant from 0 on the real number line. Each number in such a pair is called the opposite of the other. To complete the picture, 0 is defined to be its own opposite.

Page 108. In locating the opposite of a given number on the number line, you may well want to use a compass to emphasize that the number and its opposite are equidistant from 0.

Page 109. Having observed that each negative number is also the opposite of a positive number, it is apparent that we have no need for two symbolisms to denote the negative numbers. Since the lower dash "-" is applicable to all numerals for real numbers while the upper "${}^{-}$" has significance only when attached to numerals for positive numbers, we naturally retain the lower dash. There are other less important reasons for dropping the upper dash in favor of the lower: more care must be exercised in denoting negative fractions with the upper dash than with the lower; the lower dash is universally used, etc. Henceforth, then, negative numbers like ${}^{-}5$, ${}^{-}(\frac{3}{7})$, ${}^{-}\sqrt{2}$, and so on, will be written

$-5, -\frac{3}{7}, -\sqrt{2}$, etc. Thus, we read "-5" as either "negative 5" or "opposite of 5". Notice that it is not meaningful to say "$\frac{1}{2}$ equals negative negative $\frac{1}{2}$"; rather say, "$\frac{1}{2}$ equals the opposite of negative $\frac{1}{2}$", or "$\frac{1}{2}$ equals the opposite of the opposite of $\frac{1}{2}$".

The student must learn to designate the opposite of a given number by means of the definition. Do not say and do not let the student say, "To find the opposite of a number, change its sign". This is very imprecise (in fact, we have never attached a "sign" to the positive numbers) and will lead to a purely manipulative algebra which we want to avoid at all costs.

The opposite of the opposite of the opposite of a number is the opposite of that number. What is the opposite of the opposite of a negative number? The number, of course!

The student is well aware that the lower dash "-" is read "minus" in the case of subtraction. We prefer to retain the word "minus" for the operation of subtraction and not use it as an alternative word for "opposite of". Thus, the dash attached to a variable, such as "-x", will be read "opposite of".

If x is a positive number, then $-x$ is a negative number. The opposite of any negative number x is the positive number $-x$, and $-0 = 0$. Thus, the student should not jump to the conclusion that when n is a real number, then $-n$ is a negative number; this is true only when n is a positive number.

We do not like to read "-x" as "negative x". A negative number is the opposite of a positive number only. To read "-x" as "negative x" implies that x is positive, but we want pupils to think of x as any real number. Some teachers read "-x" as the negative of x. In this usage the "negative of x" is synonymous with "the opposite of x". We prefer the latter.

Answers to Problem Set 5-3a; page 110:

1. (a) The opposite of 2.3 is -2.3.
   (b) The opposite of -2.3 is 2.3.
   (c) The opposite of -(-2.3) is -2.3. Note here that the opposite of the opposite of a number is the number itself.

[pages 109-110]

(d) The opposite of $-(3.6 - 2.4)$ is $3.6 - 2.4$, or more simply, 1.2.

(e) The opposite of $-(42 \times 0)$ is (42×0) or more simply, 0.

(f) The opposite of $-(42 + 0)$ is $(42 + 0)$ or more simply, 42. Exercises (e) and (f) provide an opportunity to see whether the zero properties for addition and multiplication have lodged in the students' minds.

2. (a) If x is positive, then the opposite of x is negative.
   (b) If x is negative, then the opposite of x is positive.
   (c) If x is zero, then the opposite of x is zero.
3. (a) If the opposite of x is a positive number, then the number x itself must be negative.
   (b) If the opposite of x is a negative number, then the number itself must be positive.
   (c) If the opposite of x is 0, then the number x itself must be 0, for 0 is its own opposite.
4. (a) Since every real number has an opposite, it follows that every real number is the opposite of some real number.
   (b) Yes. See 4(a).
   (c) Every negative number is the opposite of some (positive) real number; hence, the set of negative numbers is a subset of the set of all opposites.
   (d) Some opposites, namely opposites of negative numbers, are not negative numbers. Hence, the set of opposites is not a subset of the set of negative numbers.
   (e) No. See 4(d).

Page 111. In order to motivate the "property for opposites", here it would be well to consider several other pairs of numbers; for example, a pair of distinct positive numbers, a pair of distinct negative numbers, 0 and a positive number, and 0 and a negative number.

Answers to Problem Set 5-3b; Pages 111-113.

1. Recall: One number is less than a second if it is to the left of the second on the number line. Also, a positive number is greater than a negative number.

(a) $2.97 > -2.97$.

(b) $2 > -12$.

(c) $-358 > -762$.

(d) $1 > -1$.

(e) $-121 > -370$.

(f) $0.24 > 0.12$.

(g) $0 = -0$. -0 is the opposite of 0 and they are the same point on the number line.

(h) $-0.01 > -0.1$.

(i) $0.1 > 0.01$.

2. (a) $-\frac{1}{6} < \frac{2}{7}$ and $-\frac{2}{7} < \frac{1}{6}$.

(b) $-\pi < \sqrt{2}$ and $-\sqrt{2} < \pi$.

(c) $\pi < \frac{22}{7}$ and $-\frac{22}{7} < -\pi$.

The students may need to be told that $\pi = 3.1416$ to 4 decimal places and they may determine $\frac{22}{7} = 3.1428$ to four decimal places.

(d) $3(\frac{4}{3} + 2) = 3 \times \frac{4}{3} + 3 \times 2$ by the Distributive Property

$= 4 + 6$

$= 10.$

$\frac{5}{4}(20 + 8) = (\frac{5}{4})(20) + (\frac{5}{4})8$ by the Distributive Property

$= 25 + 10$

$= 35.$

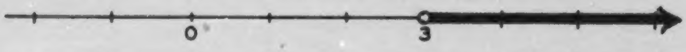
Since $10 < 35$ and $-35 < -10$, we have

$3(\frac{4}{3} + 2) < \frac{5}{4}(20 + 8)$ and $-\frac{5}{4}(20 + 8) < -3(\frac{3}{4} + 2)$.

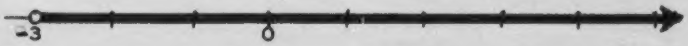
(e) $-(\frac{8 + 6}{7}) = -\frac{14}{7} = -2$.

Then $-(\frac{8 + 6}{7})$ and -2 are names for the same number.

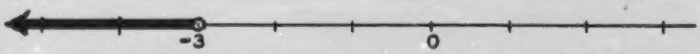
3. (a) $x > 3$.



(b) $x > -3$.



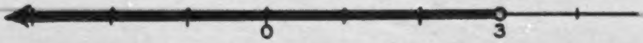
(c) $-x > 3$.



The student may use a simple trial-and-error process in the exercise above, or he may reason along some such line as this: The sentence says "opposite of x is greater than 3". Hence "the opposite of x" would describe numbers such as $\frac{13}{4}$, 4, 6.7, 10, 1000, etc. If such numbers are opposites of members of the set we are seeking, the set itself includes $-\frac{13}{4}$, -4, -6.7, -10, -1000, etc.

It would be gratifying if the students observe that the open sentences "$-x > 3$" and "$x < -3$" have the same truth set before they graph the sentences.

(d) $-x > -3$.



4. (a) The sentence states "the opposite of x is not equal to 3". There are many numbers whose opposites are not equal to 3; in fact, there is only one number whose opposite does equal 3, and this is, of course, -3. Hence the required set is the set of all real numbers except -3.

(b) $-x \neq -3$.

By the reasoning of part (a) the required set here is found to be the set of all real numbers except 3.

(c) $x < 0$.

The truth set for this sentence is the set of all real numbers less than 0, that is, the set of negative numbers.

(d) $-x < 0$.

Here the set required is the set of all real numbers whose opposites are less than zero. Now if the opposites of all members of this set are less than zero, the members of the set must be greater than zero; in other words, "$-x < 0$" and "$x > 0$" have the same truth set. Thus the truth set is the set of all positive real numbers.

(e) $-x \geq 0$.

The sentence states that the opposite of x is greater than or equal to zero, that is, that the opposite of x must be either zero or a positive number. Hence x itself must be either zero or a negative number, and the truth set is comprised of such members. Recall that a brief description of this set is the set of non-positive numbers.

(f) $-x \leq 0$.

Here the reasoning would parallel (c) above: Each member of the truth set is either zero or a positive number. This set is described briefly as the non-negative numbers.

5. A variety of answers is possible here.

(a) A is the set of all non-negative real numbers

$x \geq 0$, $-x \leq 0$, $x \nless 0$, $-x \ngtr 0$.

(b) B is the set of all real numbers not equal to -2.

$x \neq -2$, $-x \neq 2$.

Also, $x > -2$ or $x < -2$.

(c) C is the set of all real numbers not greater than -3.

$x \ngtr -3$, $-x \nless 3$.

Also, $x \leq -3$, $-x \geq 3$.

(d) $\emptyset$.

$x \geq 0$ and $x < 0$.

Also, $-x \leq 0$, $-x > 0$.

(e) E is the set of all real numbers.

$x \geq 0$ or $x < 0$.

Also, $-x \leq 0$ or $-x > 0$.

At this point the student has not performed operations of addition and multiplication with the real numbers, so these sentences should be free of such operations; or at least, he should be warned against the danger of using operations with which he is not familiar.

6. (a) $x < 1$.

(b) $-2 < x$ and $x \leq 1$.

This open sentence can be written much more suggestively as:

$-2 < x \leq 1$.

We would read this "x is greater than -2 and less than or equal to 1". This terminology emphasizes the number line picture and suggests strongly that x is 1 or is between -2 and 1.

We never write, for example "$-2 < x \geq 1$" as a shorthand for "$-2 < x$ or $x \geq 1$". The student cannot help but read "$-2 < x \geq 1$" as "x is greater than -2 and greater than or equal to 1"; in other words, he would read "$-2 < x \geq 1$" as a conjunction, when what is wanted is a disjunction.

(c) $x \leq -1$ or $x > 1$.

(d) $x > -2$ and $x < 2$ or, more briefly, $-2 < x < 2$.

7. (a) -3 is the opposite of 3. The greater is 3.

(b) -0 is the opposite of 0. They are the same number.

(c) -17 is the opposite of 17. The greater is 17.

(d) 7.2 is the opposite of -7.2. The greater is 7.2.

(e) $\sqrt{2}$ is the opposite of $-\sqrt{2}$. The greater is $\sqrt{2}$.

(f) 0.01 is the opposite of -0.01. The greater is 0.01.

(g) -2 is the opposite of -(-2). The greater is -(-2).

(h) $-(1 - \frac{1}{4})^2$ is the opposite of $(1 - \frac{1}{4})^2$. The greater is $(1 - \frac{1}{4})^2$.

(i) $-\left(1 - (\frac{1}{4})^2\right)$ is the opposite of $\left(1 - (\frac{1}{4})^2\right)$. The greater is $\left(1 - (\frac{1}{4})^2\right)$.

(j) $(\frac{1}{2} - \frac{1}{3})$ is the opposite of $-(\frac{1}{2} - \frac{1}{3})$. The greater is $(\frac{1}{2} - \frac{1}{3})$.

The opposites of (a), (d), (f), (g) and (j) may be given in terms of the opposites of the opposites, e.g.,

(a) The opposite of -7,2 is -(-7.2).

(d) The opposite of $-\sqrt{2}$ is $-(-\sqrt{2})$, etc.

*8. The relation "$\rangle$" does not have the comparison property. For example, 2 and -2 are different real numbers but neither is further from 0 than the other; in other words, none of the statements "$-2 = 2$", "$-2 \rangle 2$" and "$2 \rangle -2$" is true.

The transitive property for "$\rangle$" would read: If a, b, and c are real numbers and if $a \rangle b$ and $b \rangle c$, then $a \rangle c$. This is certainly a true statement as can be seen by substituting the phrase "is further from 0 than" for "$\rangle$" wherever it occurs.

The relations "}" and ">" have the same meaning for the numbers of arithmetic: "is further from 0 than" and "is to the right of" mean the same thing on the arithmetic number line.

9. (a) $s > -100$; s is the number representing John's score.
   (b) $n \leq 0$ and $n \geq -200$; n is the number representing my financial condition in dollars.
   (c) $d - 10 > 25$; d is the number of dollars in Paul's original debt. (Some students may observe that the variables s, n, and d would ordinarily be further restricted to be rational numbers represented by fractions whose denominators are 100.)

10. Following the hint:

$-\frac{13}{42}$ and $-\frac{15}{49}$ are to be compared.

$\frac{13}{42}\left(\frac{7}{7}\right) = \frac{91}{294}$ (Multiplication property of 1.)

$\frac{15}{49}\left(\frac{6}{6}\right) = \frac{90}{294}$

$\frac{91}{294} > \frac{90}{294}$ and $-\frac{91}{294} < -\frac{90}{294}$. Thus $-\frac{13}{42} < -\frac{15}{49}$.

In order to compare two negative rational numbers, we use the multiplication property of 1 to compare their opposites and then use the property of opposites: For real numbers a and b, if $a < b$, then $-b < -a$. We can describe this briefly as: In order to compare two negative (or two positive) rational numbers, represent them by fractions with the same denominator and compare the numbers represented by their numerators.

5-4. Absolute Value

The concept of the absolute value of a number is one of the most useful ideas in mathematics. We will find an immediate application of absolute value when we define addition and multiplication of real numbers in Chapters 6 and 7. In Chapter 9 it is used to define distance between points; in Chapter 11 we define $\sqrt{x^2}$ as $|x|$; in Chapter 13 it will provide a good example of an equation with extraneous solutions. Through Chapters 14 to 16 absolute values are involved in open sentences in two variables and in Chapter 17 it gives us interesting examples of functions. In later mathematics courses, in particular, in the calculus and in approximation theory, the idea of absolute value is indispensable.

Page 113. The usual definition of the absolute value of the real number n is that it is the number $|n|$ for which

$$|n| = \begin{cases} n, & \text{if } n \geq 0 \\ -n, & \text{if } n < 0. \end{cases}$$

This is also the form in which the absolute value is most commonly used. On the other hand, since students seem to have difficulty with definitions of this kind, we prefer to define the absolute value of a number in such a way that it can be clearly pictured on the number line. You must avoid at all costs allowing the student to think of absolute value as the number obtained by "dropping the sign". This way of thinking about absolute value, although it appears to give the correct "answer" when applied to specific numbers such as -3 or 3, leads to no end of trouble when variables are involved. Other less common names for absolute value are numerical value, magnitude, and modulus.

By observing that this "greater" of a number and its opposite is just the distance between the number and 0 on the real number line, we are able to interpret the absolute value "geometrically".

The symbolism $\sqrt{2}$ always denotes the positive number whose square is 2. The negative number whose square is 2 is written $-\sqrt{2}$.

Answers to Problem Set 5-4a; page 114:

1. (a) 7; the greater of -7 and 7 is 7.
   (b) 3; the greater of -(-3) and its opposite -3 is -(-3) or 3.
   (c) 2; 6 - 4 is another name for 2, and 2 is greater than its opposite, -2.
   (d) 0; by the multiplication property of 0, (14×0) is 0, and the absolute value of 0 is 0.

(e) 14; by the addition property of 0, (14 + 0) is 14, and 14 is greater than its opposite, -14.

(f) 3; the opposite of the opposite of the opposite of 3 is simply -3 and the opposite of -3, 3, is greater than -3.

2. (a) If x is a non-negative number, it corresponds to a point at or to the right of 0 on the real number line. Its opposite, then, is at or to the left of 0. It follows that the greater of x and its opposite $-x$ is here x. By definition, then, $|x|$ is x, a non-negative number.

(b) If x is a negative number, it corresponds to a point to the left of 0 on the real number line. Its opposite is therefore to the right of 0. Thus the greater of x and $-x$ is, in this case, $-x$; in other words, if x is a negative number $|x|$ is $-x$, the opposite of x, and thus a positive number.

(c) For every real number x, $|x|$ is a non-negative number. In parts (a) and (b) all cases, $x < 0$, $x = 0$, $x > 0$, have been considered, and in every case, $|x|$ was found to be non-negative.

3. For the negative number x, $|x|$ is greater than x since, for x negative, $|x|$ is positive by problem 2(b). Since any negative number is less than any positive number, $x < |x|$ for all negative x.

4. The set $\{-1, -2, 1, 2\}$ is closed under the operation of taking the absolute value of its elements. Taking the absolute value of each element of the set,

$$|-1| = 1$$

$$|-2| = 2$$

$$|1| = 1$$

$$|2| = 2,$$

we find that the set of absolute values of the numbers of the original set to be $\{1, 2\}$. Since $\{1, 2\}$ is a subset of $\{-2, -1, 1, 2\}$, this latter set is closed under the operation of taking absolute values of its elements.

Page 115. It is quite apparent that the greater of a positive number and its opposite is just the number itself. Furthermore, $|0|$ is defined outright to be 0. These two statements can be expressed symbolically as: If $x \geq 0$, then $|x| = x$.

For negative numbers, the number line picture should convince the student that the greater of, for example, -5, $-\frac{1}{2}$, $-3, 1$, and -467 and their opposites 5, $\frac{1}{2}$, 3.1, and 467 are, respectively, 5, $\frac{1}{2}$, 3.1, and 467. This same picture can not help but tell them that the greater of any negative number and its opposite is the opposite of the (negative) number. Symbolically, if $x < 0$, then $|x| = -x$.

We have therefore arrived at the usual definition of absolute value. For all real numbers x,

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

Answers to Problem Set 5-4b; pages 116-117:

1. (a) $|-7| < 3$ or $7 < 3$. False

   (b) $|-2| \leq |-3|$ or $2 \leq 3$. True

   (c) $|4| < |1|$ or $4 < 1$. False

   (d) $2 \not< |-3|$ or $2 \not< 3$. False

   (e) $|-5| \not< |2|$ or $5 \not< 2$. True

   (f) $-3 < 17$. True

   (g) $-2 < |-3|$ or $-2 < 3$. True

   (h) $|\sqrt{16}\,| > |-4|$ or $4 > 4$. False

   (i) $|-2|^2 = 4$ or $2^2 = 4$. True

   (b), (e), (f), (g) and (i) are true.

2. (a) $|2| + |3| = 2 + 3 = 5$.

   (b) $|-2| + |3| = 2 + 3 = 5$.

   (c) $-(|2| + |3|) = -(2 + 3) = -5$.

   (d) $-(|-2| + |3|) = -(2 + 3) = -5$.

(e) $|-7| - (7 - 5) = 7 - 2 = 5$.

(f) $7 - |-3| = 7 - 3 = 4$.

(g) $|-5| \times 2 = 5 \times 2 = 10$.

(h) $-(|-5| - 2) = -(5 - 2) = -3$.

(i) $|-3| - |2| = 3 - 2 = 1$.

(j) $|-2| + |-3| = 2 + 3 = 5$.

(k) $-(|-3| - 2) = -(3 - 2) = -1$.

(l) $-(|-2| + |-3|) = -(2 + 3) = -5$.

(m) $3 - |3 - 2| = 3 - 1 = 2$.

(n) $-(|-7| - 6) = -(7 - 6) = -1$.

(o) $|-5| \times |-2| = 5 \times 2 = 10$.

(p) $-(|-2| \times 5) = -(2 \times 5) = -10$.

(q) $-(|-5| \times |-2|) = -(5 \times 2) = -10$.

3. (a) $|x| = 1$. The truth set is $\{1, -1\}$.
(b) $|x| = 3$. The truth set is $\{3, -3\}$.
(c) $|x| + 1 = 4$. The truth set is $\{3, -3\}$, the same as that of $|x| = 3$.
*(d) $5 - |x| = 2$. The truth set is $\{3, -3\}$.

Some students may see these just by inspection. Others may think of the fact that $|x|$ is the distance on the number line from zero, so in (a), for instance, x must be 1 unit from zero. Therefore $x = 1$ or $x = -1$. Still others may reason as follows, for (d):

To find the truth set for $5 - |x| = 2$:
If $x \geq 0$, $|x| = x$, so $5 - x = 2$ or $x = 3$.
If $x < 0$, $|x| = -x$, so $5 - (-x) = 2$ or $x = -3$.
The truth set is $\{3, -3\}$.

Have several students explain their reasoning, as this is a splendid opportunity to check their understanding of absolute value.

For some better classes you may care to discuss such equations as the following. While the truth set is obvious, application of the last method above is interesting.

$$5 + |x| = 2.$$

If $x \geq 0$, $|x| = x$, so $5 + x = 2$ or $x = -3$. $x \geq 0$ and $x = -3$ gives no solution. If $x < 0$, $|x| = -x$, so $5 - x = 2$ or $x = 3$. $x < 0$ and $x = 3$ gives no solution. The truth set is $\emptyset$.

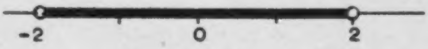
4. Here students may have difficulty in finding a starting point. It may be helpful for them to refer back to Problem Set 5-4a, Problems 2 and 3.
   (a) $|x| \geq 0$ is true for all real numbers x.
   If $x \geq 0$, $|x| \geq 0$. See Problem Set 5-4a, Problem 2(b);
   If $x < 0$, $|x| > 0$. See Problem Set 5-4a, Problem 3(b).
   (b) $x \leq |x|$ is true for all real numbers x.
   If $x \geq 0$, $x = |x|$. If $x < 0$, $x < |x|$.
   (c) $-x \leq |x|$ is true for all real numbers x.
   If $x \geq 0$, $-x \leq |x|$. If $x < 0$, $-x = |x|$.
   (d) $-|x| \leq x$ is true for all real numbers x.
   If $x \geq 0$, $-|x| \leq x$. If $x < 0$, $-|x| = x$.
5. If m is the number of dollars which I have and j is the number of dollars which John has, the first part of the first sentence of the problem may be translated from "John has less money than I" to $j < m$, and the second part of the sentence, "I have less than \$20", to $m < 20$. Now the two sentences $j < m$ and $m < 20$ are available. Using the transitive property, we have

$$j < m < 20,$$

$$j < 20;$$

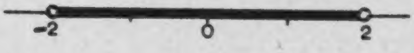
in other words, John has less than \$20.

6. Graph the truth sets of the following sentences:
   (a) $|x| < 2$.

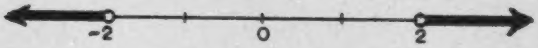


The student may arrive at the required graph by trial of different numbers for x in the sentence. He may instead reason the exercise out somewhat as follows: The sentence states that "The absolute value of x is less than 2". On the number line, this statement becomes "x is less than 2 units away from 0". Therefore, the graph of "$|x| < 2$" is the one given above.

(b) $x > -2$ and $x < 2$.

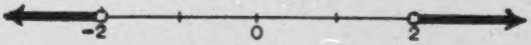


(c) $|x| > 2$.



As in part (a) the student here may find the required set by trial-and-error, or by recalling the interpretation of absolute value as a distance on the number line as in (a) above.

(d) $x < -2$ or $x > 2$.



7. The graphs of the sentences in 6(a) and 6(b) are the same. The graphs of the setnences in 6(c) and 6(d) are the same. The student should begin to see that the statements (a) and (b) say the same thing, as do (c) and (d).

8. If x is negative, the absolute value of x, being the greater of the number and its opposite, is the opposite of x, that is

$$|x| = -x \quad \text{or} \quad -x = |x|.$$

Since $-x$ and $|x|$ are in this case names for the same number, their opposites also will be names for the same number, so that $-(-x) = -|x|$. As the opposite of the opposite of x is x, we may say $-(-x) = x$ and finally

$$x = -|x|.$$

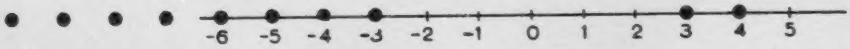
9. The set of integers less than 5 is the set

$$\{. \; . \; ., -1, 0, 1, 2, 3, 4\}.$$

The set of integers less than 5 whose absolute values are greater than 2 is

$$\{. \; . \; ., -5, -4, -3, 3, 4\}.$$

-5 and -10 are both elements of this set, but 0 is not.



10. Three numbers:
   (a) In P but not in I: All positive numbers except integers, $\frac{3}{10}$, $\sqrt{2}$, π, 5.3, etc.
   (b) In R but not in P: All non-positive real numbers, -7, $-\pi$, 0, $-\sqrt{2}$, $-\frac{5}{3}$, etc.
   (c) In R but not in P or I: All non-positive real numbers, except integers,
   $-\frac{17}{5}$, -2.74, $-\frac{\pi}{2}$, $-\sqrt{2}$, etc.
   (d) In P but not in R: All non-real positive numbers. Since there are none, this is the empty set, $\emptyset$.

11. The number representing the degrees of temperature is t. (In this exercise we interpret "within" to exclude the end points -5 and 5.) Then

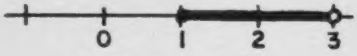
$$|t| < 5$$

or $t > -5$ and $t < 5$ or $-5 < t < 5$.

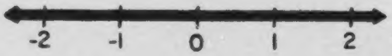
12. $|x| = 0$ has the truth set $\{0\}$.
$|x| = -1$ has the empty set $\emptyset$ as its truth set. Students should recall the difference between $\{0\}$ and $\emptyset$.

Answers to Review Problems; pages 119-120:

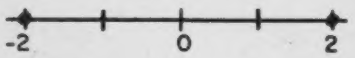
1. Sentences (b), (d), (e), (f) are true.
2. Sentences (a), (d), (f) are false.
3. (a)



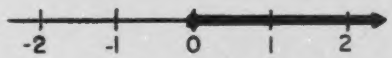
(c)



(b)

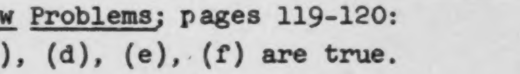


(d)

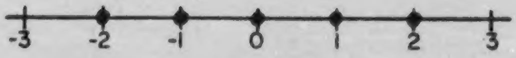


4. (a) $\emptyset$.
   (b) The set of all real numbers.
   (c) The set of all real numbers greater than -3 and less than 2.
   (d) The set of all non-positive numbers.
   (e) $\emptyset$.
   (f) The set of all real numbers

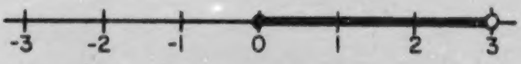
5. (a)



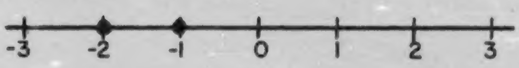
(b)



(c)



(d)

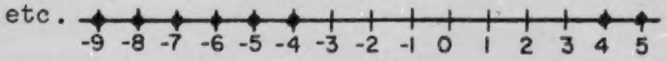


6. (a) If t is the number of degrees of the average temperature on Thursday, then $t < (-10) - 4$; or, if f is the number of degrees of the average temperature on Friday, and t is as above, then $t < f - 4$, and, since f is -10, $t < (-10) - 4$.

(b) If s is the number of degrees of the average temperature, and if we interpret "within" to exclude the end points -11 and 1, then this can be written as a compound open sentence: $s > -11$ and $s < 1$; or $-11 < s < 1$; or $|s + 5| < 6$.

7. (a), (b), and (e) are true statements.

8.

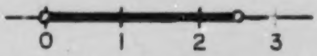


9. If n is the integer, $n + 1$ is its successor, and

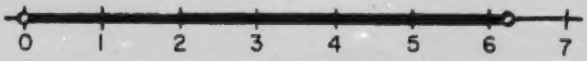
$$n + (n + 1) = n + 1$$

whose truth set is $\{0\}$.

10. (a) If s is the number of units in the side of this square, s is positive and $4s$ is the perimeter of the square. A sentence for this is $s > 0$ and $4s < 10$.



(b) If A is the number of units in the area of the square, then $A = s^2$, where $s > 0$ and $4s < 10$ as in part (a). Since A is s^2, and s is a number from the set of numbers between 0 and 2.5, the truth set of A is the set of numbers between 0 and 6.25.



CHAPTER 5

Suggested Test Items

1. Determine which of the following sentences are true:

   (a) $-\frac{2}{3} \neq \frac{2}{3}$

   (b) $|-8| < 8$

   (c) $|2| \geq |-2|$

   (d) $(-1) = -\big(-(-1)\big)$

   (e) $-(|-5| + |-7|) = 12$.

2. From R, the set of all real numbers, describe three infinite sets A, B, C each a subset of R; choose C such that it is a subset of A but not of B.

3. Rearrange the following numbers in order from the least to the greatest:

$$-\frac{1}{8},\ -2,\ -\frac{1}{3},\ 0,\ \frac{1}{2},\ -\frac{5}{2}.$$

4. Which is the lesser of the two?

   (a) $2,\ -|-3|$

   (b) $-4,\ -7$

   (c) $|-5|,\ 2$

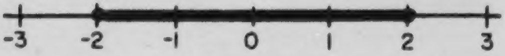
   (d) $-\frac{13}{11},\ -\frac{13}{12}$

5. If $a < b$, write an open sentence expressing the order of $-a$ and $-b$.

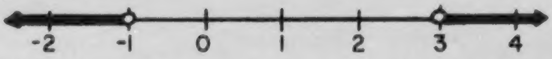
6. If $a < b$, is it possible to write an open sentence expressing the order of $-a$ and b? Explain.

7. Write an open sentence whose graph is:

   (a)



   (b)



8. If b is a negative number, classify each as the following: (Positive, negative)

   (a) $-b$

   (b) $|b|$

   (c) $|-b|$

   (d) $-|-b|$

   (e) $-(-b)$

   (f) $-\big|-|-b|\big|$.

9. If is is known that $-17 = -\frac{612}{36}$ and $-17 = -\frac{323}{19}$, explain how this information may be used to decide which is the greater of $-\frac{611}{36}$ and $-\frac{324}{19}$.

10. Draw the graph of the truth set of each of the open sentences:
    - (a) $|x| = 3$
    - (b) $|x| - 1 = 5$
    - (c) $|x| = 0$
    - (d) $|x| < 0$
    - (e) $x < 4$
    - (f) $-|x| < 0$
11. Describe the truth set of each of the open sentences:
    - (a) $|x| > x$
    - (b) $|x| < x$
    - (c) $|x| = x$
    - (d) $|x| = -x$
12. Describe the variable and translate into an open sentence:
    - (a) Peter lives closer to school than Ralph. Peter is more than $3\frac{1}{2}$ miles from school. What distance is the school from Ralph's home?
    - (b) Henry's score is i and Joe's score is j, and Henry's score is at least 10 higher than Joe's.
13. (a) If $x > r$ and $x < n$ what is the relation between r and n? Can the transitive property be used here?
14. Is the absolute value of x always x. Why? Why not?

Chapter 6

PROPERTIES OF ADDITION

One of our main objectives in this course is to study the structure of the real number system. We began our study of the set of all real numbers in Chapter 5. However, a system of numbers is a set of numbers and the operations on these numbers. Hence, we do not really have the real number system until we define the operations of addition and multiplication for negative numbers.

The aim of this and the next two chapters is to correct this deficiency. Our point of view is that the operations must be extended from the non-negative reals to all real numbers. Thus the definitions of addition and multiplication for all real numbers must be formulated exclusively in terms of the non-negative numbers and operations (including oppositing) on them. We, of course, insist on preserving the fundamental properties of the operations.

The present chapter is concerned with addition. We first consider some examples using gains and losses to suggest how addition involving negative numbers ought to be defined. The number line is also used to picture this, and finally a precise definition is formulated, first in English and then in the language of algebra.

Chapter 7 is concerned with multiplication. It is more difficult to find "real life situations" which will suggest what multiplication involving negative numbers ought to be. However, after we have addition, insistence on the distributive property suggests how multiplication must be defined.

Order in the real numbers was introduced in Chapter 5. In Chapter 8, we return to order and obtain its properties with respect to addition and multiplication. There is an important shift in our point of view on order in this chapter. Previously we have tended to use order as a convenient way to discuss certain properties of real numbers. In this sense "<" or ">" were hardly more than fragments of our language. In Chapter 8, we treat "<" as an order relation. A similar shift in point of view had to be made earlier in the case of addition, for example. In arithmetic, the sign "+" in the expression "25 + 38" is nothing more than a reminder or command to carry out a previously learned process to obtain "63". The idea of "+" as an operation to be studied for its own sake is quite a different notion of addition from that in arithmetic. Thus, in Chapter 8, the order relation becomes a mathematical object in its own right.

At the end of Chapter 8 there is an extensive summary in which we bring together the various properties of the real number system which we have obtained so far. An attempt is made here to begin thinking of the real number system from the deductive point of view. In other words, it is an undefined set of elements endowed with an operation of addition, an operation of multiplication and an order relation subject to certain assumed properties from which all other properties can be deduced by proofs.

Very quickly in the present chapter the student should learn how to find sums involving negative numbers. This is easy and is suggested completely by the profit and loss examples. However, our immediate objective is more ambitious than just to teach the arithmetic of negative numbers. We want to bring out the important fact that what is really involved here is an extension of the operation of addition from the numbers of arithmetic (where the operation is familiar) to all real numbers in such a way that the basic properties of addition are preserved. This means that we must give a definition of addition in terms of only non-negative numbers and familiar operations on them. The result in the language of algebra is a formula for $a + b$ involving the familiar operations of addition, subtraction and opposite applied to the non-negative $|a|$ and $|b|$. The complete formula appears formidable because of the variety of cases. However the idea is simple and is nothing more than a general statement of exactly what we always do in obtaining the sum of negative numbers.

The main problem is to lead up to the general definition of $a + b$ in a plausible way. We have chosen to make full use of the number line and especially to make use of the interpretation of absolute value as a distance from zero. Therefore, in Section 6-1, the new $a + b$, as suggested by profit and loss, is first pictured on the number line. It is obtained by moving a distance $|b|$ to the right of a if b is positive and to the left of a if b is negative. In Section 6-2 the case of the sum of two negative numbers is considered in some detail first. Then the other cases are considered more briefly leading up to the general definition first in English and then in the language of algebra.

The comments at the beginning of Chapter 5 about former S.M.S.G. students apply to this chapter too. In addition we note that the 8th Grade S.M.S.G. material has given them the addition property of equality and the multiplication property of equality. They should find these ideas rather easy when they meet them in this chapter, but do not let the students as a consequence become too mechanical in their work with equations.

A reference for the extension of the operations from the numbers of arithmetic to the real numbers is Haag, *Studies in Mathematics*, Volume III, *Structure of Elementary Algebra*, Chapter 3, Section 4.

6-1. Addition of Real Numbers

The profit and loss approach to adding positive and negative numbers seems to be a natural one. The only thing which may seem new to the student is writing it down in terms of positive and negative numbers.

Page 122. If we add 0 no motion results.

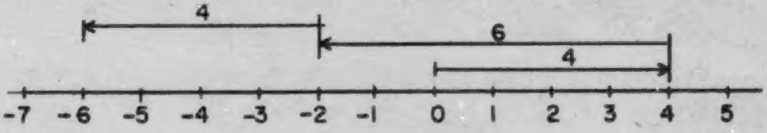
Answers to Problem Set 6-1; page 123:

If your students have mastered the arithmetic of negative numbers; that is, if they have no trouble finding sums such as $(-7) + 5$, then Problem 1 could be omitted.

1. (a) $(-6) + 8 = 2$. Two yards were gained.
   (b) $(-60) + 50 = -10$. John had a net loss of 10¢.
   (c) $(-15) + 10 = -5$. Five degrees below 0.
   $(-15) + 30 = 15$. Fifteen degrees above 0.

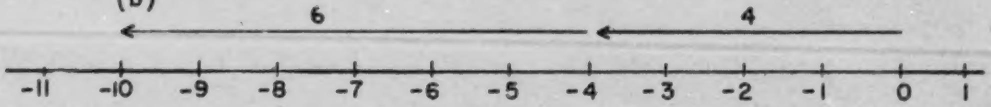
(d) (-6) + (-3) + 4 + 5 = 0. The net gain was 0 pounds.

2. (a)

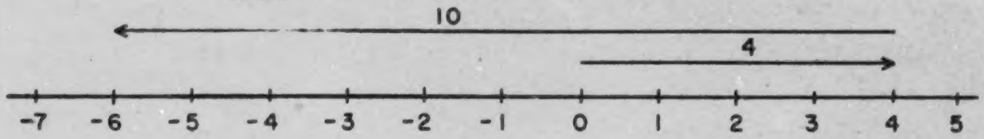

4
6
4
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5

The sum is -6

(b)

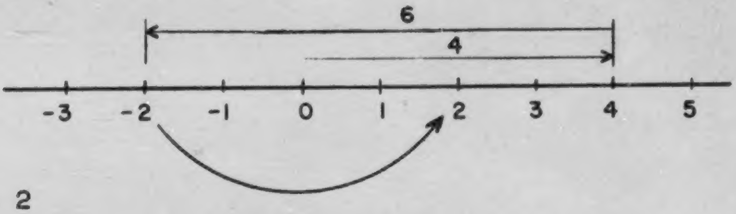

6
4
-11
-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1

then


10
4
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5

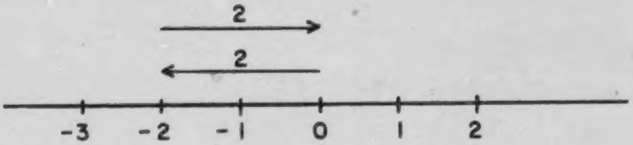
The sum is -6

(c)

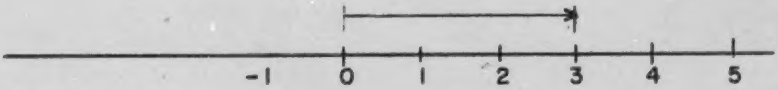

6
4
-3
-2
-1
0
1
2
3
4
5

The sum is 2

(d)

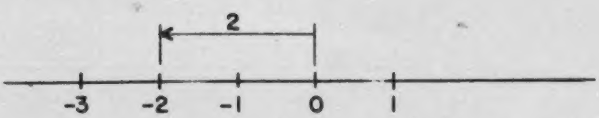


then

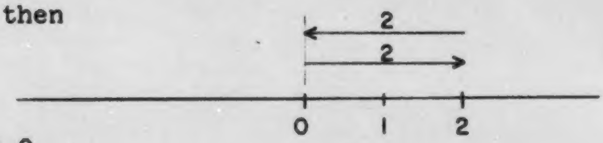


The sum is 3

(e)

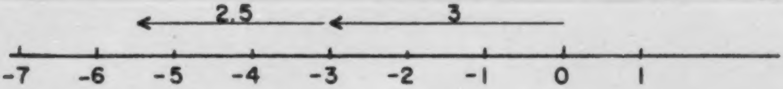


then



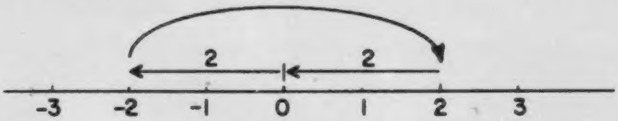
The sum is 0

(f)


2.5
3
-7
-6
-5
-4
-3
-2
-1
0
1

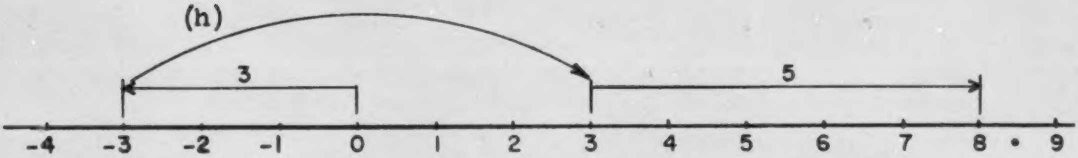
The sum is -5.5

(g)

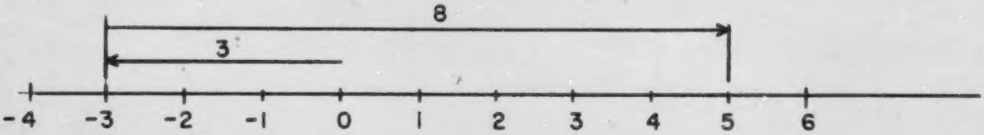


The sum is 0

(h)


3
5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9

then


8
3
-4
-3
-2
-1
0
1
2
3
4
5
6

The sum is 5

3. (a) Move from 0 to 7 on the number line, then move 10 units to the right.
   (b) Move from 0 to 7 on the number line, then move 10 units to the left.
   (c) Move from 0 to 10 on the number line, then move 7 units to the left.
   (d) Move from 0 to -10 on the number line, then move 7

units to the left.

(e) Move from 0 to 10 on the number line, then move 7 units to the right.

(f) Move from 0 to -7 on the number line, then move 10 units to the left.

(g) Move from 0 to -7 on the number line, then move 10 units to the right.

(h) Move from 0 to -10 on the number line, then move 7 units to the right.

(i) Move from 0 to -10 on the number line, then move 0 units.

(j) Move from 0 to 0 on the number line, then move 7 units to the right.

4. In (a), (e) and (j).

5. When both numbers were negative, the sum was negative, and was the opposite of the sum obtained when both numbers were positive.

Answers to Problem Set 6-2a; pages 125-126:

1. (a) $$\begin{aligned}(-2) + (-7) &= -(|-2| + |-7|)\\ &= -(2 + 7)\\ &= -9\end{aligned}$$

A loss of \$2 followed by a loss of \$7 is a net loss of \$9.

(b) $$\begin{aligned}(-4.6) + (-1.6) &= -(|-4.6| + |-1.6|)\\ &= -(4.6 + 1.6)\\ &= -6.2\end{aligned}$$

Move from 0 to -4.6 on the number line, then move 1.6 units to the left. You arrive at -6.2.

(c) $$\begin{aligned}(-3\tfrac{1}{3}) + (-2\tfrac{2}{3}) &= -(|-3\tfrac{1}{3}| + |-2\tfrac{2}{3}|)\\ &= -(3\tfrac{1}{3} + 2\tfrac{2}{3})\\ &= -6\end{aligned}$$

Move from 0 to $-3\frac{1}{3}$ on the number line, then move $2\frac{2}{3}$ units to the left. You arrive at -6.

[page 125]

(d) $$\begin{aligned}(-25) + (-73) &= -(|-25| + |-73|)\\ &= -(25 + 73)\\ &= -98\end{aligned}$$

A loss of \$25 followed by a loss of \$73 is a net loss of \$98.

(e) $5\frac{1}{2} + 2\frac{1}{2} = 8$

Here we have a problem involving only the addition of positive numbers, so that the definition for the addition of negative numbers cannot be used.

2. (a) $(-6) + (-7) = -13$

(b) $(-7) + (-6) = -13$

(c) $$\begin{aligned}-(|-7| + |-6|) &= -(7 + 6)\\ &= -13\end{aligned}$$

(d) $6 + (-4) = 2$

(e) $(-4) + 6 = 2$

(f) $$\begin{aligned}|6| - |-4| &= 6 - 4\\ &= 2\end{aligned}$$

(g) $0 + (-3) = -3$

(h) $$\begin{aligned}-(|-3| - |0|) &= -(3 - 0)\\ &= -3\end{aligned}$$

(i) $$\begin{aligned}3 + \big((-2) + 2\big) &= 3 + 0\\ &= 3\end{aligned}$$

3. (a) $\{-3\}$

(b) $\{-3\}$

(c) $\{-3\}$

(d) $\{-3\}$

If the definition of the sum of two negative numbers is used, the truth sets of the sentences are easily found.

4. The distance from 0 is (the absolute value of) the difference between the absolute values of the numbers.

5. From the point of view of the number line: If the distance moved to the right was greater than the distance moved to the left, the sum was positive; if the distance moved to the left was greater, the sum was negative.

In terms of absolute value: If the negative number has the greater absolute value, the sum is negative; if the positive number has the greater absolute value, the sum is positive.

*6. The sentence is true for all non-negative values of x since: If x is positive, $-x$ is negative, and our definition applies.

If x is 0, we have: $(-1) + 0 = -(|-1| + 0)$

$= -1.$

If the domain of x is extended to the set of all real numbers, the sentence is not true; for if x is negative, $-x$ is positive, and our definition for the sum of two negative numbers does not apply.

Answers to Problem Set 6-2b; pages 127-129:

1. (a) $(-5) + 3 = -(|5| - |3|) = -(5 - 3) = -2$

(b) $(-11) + (-5) = -(|-11| + |-5|) = -(11 + 5) = -16$

(c) $(-\frac{8}{3}) + 0 = -(|-\frac{8}{3}| - |0|) = -(\frac{8}{3} - 0) = -\frac{8}{3}$

(d) $2 + (-2) = |2| - |-2| = 2 - 2 = 0$

(e) $18 + (-14) = |18| - |-14| = 18 - 14 = 4$

(f) $12 + 7.4 = 19.4$

(g) $(-\frac{2}{3}) + 5 = |5| - |-\frac{2}{3}| = 5 - \frac{2}{3} = \frac{15}{3} - \frac{2}{3} = \frac{13}{3}$

[pages 127-129]

(h) $(-35) + (-65) = -(|-35| + |-65|)$
$= -(35 + 65)$
$= -100$

2. Since the sum of two real numbers is a real number, the set of all real numbers is closed under addition.

3. Since the sum of two negative real numbers is a negative real number, the set of all negative real numbers is closed under addition.

4. The daily mean temperatures were:

$71 + (-7)$, or 64
$71 + 2$, or 73
$71 + (-3)$, or 68
$71 + 0$, or 71
$71 + 9$, or 80
$71 + 12$, or 83
$71 + (-6)$, or 65

The sum of the variations is

$$(-7) + 2 + (-3) + 0 + 9 + 12 + (-6), \text{ or } 7.$$

5. (a) If x is 5, then $5 + 2 = 7$ is true.
(b) If y is -10, then $3 + (-10) = -7$ is true.
(c) If a is -5, then $(-5) + 5 = 0$ is true.
(d) If b is 10, then $10 + (-7) = 3$ is true.
(e) If x is 0, then $(-\frac{5}{6}) + 0 = -\frac{5}{6}$ is true.
(f) If c is -4, then $(-4) + (-3) = -7$ is true.
(g) If y is $-\frac{9}{6}$, then $(-\frac{9}{6}) + \frac{2}{3} = -\frac{5}{6}$ is true.
(h) If x is 20, then $\frac{1}{2}(20) + (-4) = 6$ is true.
(i) If y is 3, then $\big(3 + (-2)\big) + 2 = 3$ is true.
(j) If x is (-1), then $\big(3 + (-1)\big) + (-3) = -1$ is true.

6. (a) false
(b) true
(c) true
(d) true
(e) false
(f) false
(g) false
(h) true
(i) true

7. (a) If x is the distance from the starting point (with north taken as the positive direction), then

$$x = 40 + (-55).$$

(b) If n is the third number, then

$$(-9) + 28 + n = (-52).$$

(c) If c is the temperature change between 4 P.M. and 8 P.M., then

$$-2 + 15 + 6 + c = -9.$$

(d) If g is the number of pounds gained the third week, then

$$200 + (-4) + (-6) + g = 195.$$

(e) If s is the number of points change in the stock price listing,

$$83 + (-5) + s = 86.$$

6-3. Properties of Addition

We have seen in Section 6-1 that the definition of addition of real numbers satisfies two of three requirements we make. It includes as a special case the familiar addition of numbers of arithmetic, and it agrees with our intuitive feeling for this operation as shown in working with gains and losses and with the number line. The third requirement is that addition of real numbers have the same basic properties that we observed for addition of numbers of arithmetic. It would be awkward, for instance, to have addition of numbers of arithmetic commutative and addition of real numbers not commutative.

Notice that, while we did not call them such for the students, the commutative and associative properties were, for all intents and purposes, regarded as axioms for the numbers of arithmetic, and the operation of addition was regarded essentially as an undefined operation. For the real numbers, however, we have made a definition of addition in terms of earlier concepts. If our definition has been properly chosen, we should find that the properties can be proved as theorems. While many of the students will not fully appreciate all this, you should have it in mind as background.

We have tried to give the students a feeling for the provability of these properties, but very few of them will be ready to follow through the details. However, for the occasional student who is able and interested, we have left the way open for him to satisfy himself fully that the properties hold in all cases, not just in some particular cases he might try.

Page 129. Perhaps you will wish to remind the students of the closure property: For any real numbers a and b, $a + b$ is a unique real number. This property is included in the summary at the end of Chapter 8.

Three cases of addition of real numbers have been illustrated. Besides the familiar case of the addition of two positive numbers, there remain two cases to be illustrated. They are:

(1) If $a \geq 0$ and $b < 0$

$$a + b = -(|b| - |a|), \text{ if } |b| > |a|$$

e.g.

$$5 + (-8) = (-8) + 5$$

(2) If $b \geq 0$ and $a < 0$

$$a + b = -(|a| - |b|), \text{ if } |a| > |b|$$

e.g.

$$(-8) + 5 = 5 + (-8)$$

Pages 129-130. It is probably not a good idea to try the proof of commutativity in class and proof of associativity certainly should not be attempted. There are too many cases and it is difficult to consider them systematically. However, some good students like this kind of thing and may enjoy trying their hand at it. The eager student who wants to write a general demonstration of the commutative property of addition of real numbers for all cases might write as follows.

To show that $a + b = b + a$ for all real numbers a and b:

If $a \geq 0$ and $b \geq 0$, $a + b = b + a$ because they are numbers of arithmetic.

If $a < 0$ and $b < 0$,
$$a + b = -(|a| + |b|)$$
$$b + a = -(|b| + |a|)$$

But $|a|$ and $|b|$ are numbers of arithmetic,
so $|a| + |b| = |b| + |a|$
Therefore $a + b = b + a$

If $a \geq 0$ and $b < 0$,

$$\left.\begin{aligned} a + b &= (|a| - |b|) \\ b + a &= (|a| - |b|) \end{aligned}\right\} \text{ if } |a| \geq |b|$$

$$\left.\begin{aligned} a + b &= -(|b| - |a|) \\ b + a &= -(|b| - |a|) \end{aligned}\right\} \text{ if } |b| \geq |a|$$

In either case $a + b = b + a$, since opposites of equals are equal.

If $a < 0$ and $b \geq 0$,

$$\left.\begin{aligned} a + b &= -(|a| - |b|) \\ b + a &= -(|a| - |b|) \end{aligned}\right\} \text{ if } |a| \geq |b|$$

$$\left.\begin{aligned} a + b &= (|b| - |a|) \\ b + a &= (|b| - |a|) \end{aligned}\right\} \text{ if } |b| \geq |a|$$

In either case $a + b = b + a$.

A careful examination will show that we have considered every possible case, and every time we found

$$a + b = b + a.$$

Therefore this is true for all real numbers a and b.

Page 130 (line 10). Each pair of numerals names one number.

In the case of the associative property of addition, it is unlikely that even a good student should be encouraged to spend his time trying to demonstrate all cases. The proof is a tedious affair because of the number of cases which have to be considered. To compute $(a + b) + c$, for example, one must examine eighteen cases:

(1) $a \geq 0, \quad b \geq 0, \quad c \geq 0$
(2) $a < 0, \quad b < 0, \quad c < 0$
(3) $a \geq 0, \quad b \geq 0, \quad c < 0, \quad a + b \geq |c|$
(4) $a \geq 0, \quad b \geq 0, \quad c < 0, \quad a + b < |c|$
(5) $a < 0, \quad b < 0, \quad c \geq 0, \quad |a + b| \geq c$
(6) $a < 0, \quad b < 0, \quad c \geq 0, \quad |a + b| < c$
(7) $a \geq 0, \quad b < 0, \quad c \geq 0, \quad a \geq |b|$
(8) $a \geq 0, \quad b < 0, \quad c \geq 0, \quad a < |b|, \quad |a + b| \geq c$
(9) $a \geq 0, \quad b < 0, \quad c \geq 0, \quad a < |b|, \quad |a + b| < c$
(10) $a < 0, \quad b \geq 0, \quad c \geq 0, \quad |a| \geq b, \quad |a + b| \geq c$

(11) $a < 0, \quad b \geq 0, \quad c \geq 0, \quad |a| \geq b, \quad |a + b| < c$
(12) $a < 0, \quad b \geq 0, \quad c \geq 0, \quad |a| < b,$
(13) $a \geq 0, \quad b < 0, \quad c < 0, \quad a \geq |b|, \quad a + b \geq |c|$
(14) $a \geq 0, \quad b < 0, \quad c < 0, \quad a \geq |b|, \quad a + b < |c|$
(15) $a \geq 0, \quad b < 0, \quad c < 0, \quad a < |b|$
(16) $a < 0, \quad b \geq 0, \quad c < 0, \quad |a| \geq b$
(17) $a < 0, \quad b \geq 0, \quad c < 0, \quad |a| < b, \quad a + b \geq |c|$
(18) $a < 0, \quad b \geq 0, \quad c < 0, \quad |a| < b, \quad a + b < c$

In addition to (1) and (2) above, there are sixteen more cases for $a + (b + c)$! !

If the student is persistent, have him list the cases for $(a + b) + c$ as above. Such an examination of the possibilities would be just as valuable as a proof; if he could set forth all the cases above, he would certainly understand addition of real numbers.

Page 131. The Addition Property of Opposites says that the sum of a and $(-a)$ is zero. It does not say that if the sum of a and another number is zero, the other number is $(-a)$. This fact is proved later.

Answers to Problem Set 6-3; pages 131-132:

1. (a) The left numeral is

$\left(3 + (-3) + 4\right) = \left(3 + (-3)\right) + 4$ associative property of addition.

$= 0 + 4.$ addition property of opposites.

The right numeral is

$0 + 4.$

(b) The right numeral is

$\left((-3) + 5\right) + 7 = \left(5 + (-3)\right) + 7.$ commutative property of addition.

The left numeral is

$\left(5 + (-3)\right) + 7.$

(c) The left numeral is

$\left(7 + (-7)\right) + 6 = 0 + 6.$ addition prop-

	erty of opposites.
$= 6$	addition property of 0.

The right numeral is 6.

(d) The left numeral is

$\lvert -1 \rvert + \lvert -3 \rvert + (-3) = 1 + 3 + (-3)$	definition of absolute value
$= 1 + \big(3 + (-3)\big)$	associative property of addition.
$= 1 + 0.$	addition property of opposites.
$= 1$	addition property of 0.

The right numeral is 1.

(e) The right numeral is

$\big((-2) + 3\big) + (-4) = (-2) + \big(3 + (-4)\big)$	associative property of addition.

The left numeral is

$$(-2) + \big(3 + (-4)\big).$$

(f) The left numeral is

$(-\lvert -5 \rvert) + 6 = (-5) + 6$	definition of absolute value
$= 6 + (-5).$	commutative property of addition.

The right numeral is

$$6 + (-5).$$

2. (a) $\frac{5}{16} + 28 + (-\frac{5}{16}) = \left(\frac{5}{16} + (-\frac{5}{16})\right) + 28$

$= 28$

(b) $.27 + (-18) + 3 + .73 = (.27 + .73) + \big((-18) + 3\big)$

$= 1 + (-15)$

$= -14$

(c) $$\begin{aligned}(-5) + 32 + 3 + (-8) &= (-5) + \big(3 + (-8)\big) + 32\\ &= (-5) + (-5) + 32\\ &= (-10) + 32\\ &= 22\end{aligned}$$

(d) $(-\frac{1}{2}) + 7 + (-2) + (-\frac{3}{2}) + 2$

$$\begin{aligned}&= \big((-\tfrac{1}{2}) + (-\tfrac{3}{2})\big) + \big((-2) + 2\big) + 7\\ &= (-2) + 0 + 7\\ &= 5\end{aligned}$$

(e) $\frac{5}{3} + (-3) + 6 + \frac{1}{3} + (-2)$

$$\begin{aligned}&= \big((\tfrac{5}{3} + \tfrac{1}{3}) + (-2)\big) + \big((-3) + 6\big)\\ &= \big(2 + (-2)\big) + 3\\ &= 0 + 3\\ &= 3\end{aligned}$$

(f) $253 + (-67) + (-82) + (-133)$

$$\begin{aligned}&= 253 + \big((-67) + (-133)\big) + (-82)\\ &= 253 + \big((-200) + (-82)\big)\\ &= 253 + (-282)\\ &= -29\end{aligned}$$

(g) $$\begin{aligned}|-\tfrac{3}{2}| + \tfrac{5}{2} + (-7) + |-4| &= \big((\tfrac{3}{2} + \tfrac{5}{2}) + 4\big) + (-7)\\ &= 8 - 7\\ &= 1\end{aligned}$$

(h) $$\begin{aligned}(x + 2) + (-x) + (-3) &= \big(x + (-x)\big) + \big(2 + (-3)\big)\\ &= 0 + (-1)\\ &= -1\end{aligned}$$

(i) $w + (w + 2) + (-w) + 1 + (-3)$

$$\begin{aligned}&= w + \big(w + (-w)\big) + \big((2 + 1) + (-3)\big)\\ &= w + 0 + 0\\ &= w\end{aligned}$$

3. (a)
$$\begin{aligned} x &= x + (-x) + 3 \\ x &= \big(x + (-x)\big) + 3 \\ x &= 0 + 3 \\ x &= 3 \end{aligned}$$

(b)
$$\begin{aligned} m + 7 + (-m) &= m \\ m + (-m) + 7 &= m \\ \big(m + (-m)\big) + 7 &= m \\ 0 + 7 &= m \\ 7 &= m \end{aligned}$$

(c)
$$\begin{aligned} n + (n + 2) + (-n) + 1 + (-3) &= 0 \\ n + (-n) + (n + 2) + 1 + (-3) &= 0 \\ \big(n + (-n)\big) + n + (2 + 1) + (-3) &= 0 \\ 0 + n + \big(3 + (-3)\big) &= 0 \\ n + 0 &= 0 \\ n &= 0 \end{aligned}$$

(d)
$$\begin{aligned} (y + 4) + (-4) &= 9 + (-4) \\ y + \big(4 + (-4)\big) &= 9 + (-4) \\ y + 0 &= 9 + (-4) \\ y &= 9 + (-4) \\ y &= 5 \end{aligned}$$

*4. If $a < 0$ and $b < 0$, then

$a + b = -(\lvert a\rvert + \lvert b\rvert)$	definition of addition
$= -(\lvert b\rvert + \lvert a\rvert)$	commutative property of addition for numbers of arithmetic
$= b + a$	definition of addition

*5. If $a \geq 0$, then

$a + 0 = a.$	definition of addition

If $a < 0$, then

$a + 0 = -(\lvert a\rvert - \lvert 0\rvert)$	definition of addition
$= -(\lvert a\rvert)$	subtraction as in arithmetic
$= -(-a)$	definition of absolute value of a negative number.
$= a.$	for all a, the opposite of the opposite of a is a.

[page 132]

*6. If $a = 0$,	
$a + (-a) = 0 + (-0)$	
$= 0 + 0$	opposite of 0 is 0.
$= 0$	addition property of 0
If $a \neq 0$,	
either a is positive or	comparison property
a is negative.	
If a is positive,	definition of opposites
$-a$ is negative.	
If a is negative,	
$-a$ is positive.	
$\lvert a \rvert = \lvert -a \rvert$	definition of absolute value.
Hence, $a + (-a) = 0$	definition of addition

6-4. The Addition Property of Equality

You may recognize the "Addition Property of Equality" as the traditional statement, "If equals are added to equals, the sums are equal." While we shall have frequent occasion to use this idea, we prefer not to treat it as a property of real numbers because it is really just an outgrowth of two names for the same number. The name "Addition Property of Equality" will be a convenient way to refer to this idea when we need to use it.

From another point of view, the addition property of equality can also be thought of as being a way of saying that the operation of addition is single valued; that is, the result of adding two given numbers is a single number. In other words, whenever we add two given numbers we always get the same result. Therefore, if a, b and c are real numbers and $a = b$, then the statement "$a + c = b + c$" can be thought of as saying that the result of adding the two given numbers was the same when they had the names "a" and "c" as when they had the names "b" and "c".

Page 133. While we indicated the addition on the right, the same property would hold of course for addition on the left,

$$\text{If } a = b, \text{ then } c + a = c + b.$$

We shall be free to use this property for addition on either side.

Page 134. Later in Chapters 7 and 13 we shall learn about equivalent equations and the permissible operations which keep equations equivalent. For the present, however, notice that all we are claiming when we apply the addition property of equality is that *if* a number makes the original equation true, it will make the new equation true. We then have a chance to test each number of the truth set of the new equation and see whether it makes the original equation true. It is necessary to make this check every time, until we have the more complete reasoning of Chapter 7. We recommend the form shown in Example 3. When the student is more familiar with the addition property of equality he may be encouraged to *think* of adding $\frac{1}{2}$ to both sides and not write that step. We must insist, however, that the phrase, "If the equation is true, for some number x, then," be written each time, thereby emphasizing the real meaning of what we are doing. We shall see some problems where there is *no* x which makes the sentence true, and this shows the need for caution.

Answers to Problem Set 6-4; page 135:

The pupil should be able to give a reason for each step in solving an equation. You may want him to write them down; if so, you will probably want to suggest a shorthand for doing this.

1. If $x + 5 = 13$ is true for some x,
   then $x + 5 + (-5) = 13 + (-5)$ is true for the same x,
   $x + 0 = 8$ is true for the same x,
   $x = 8$ is true for the same x.
   If $x = 8$,
   the left member is $8 + 5 = 13$;
   the right member is 13.
   Hence, the truth set is $\{8\}$;
   the only solution is 8.

2. If $(-6) + 7 = (-8) + x$ is true for some x,
then $8 + (-6) + 7 = 8 + (-8) + x$ is true for the same x,
$9 = 0 + x$ is true for the same x,
$9 = x$ is true for the same x.
If $x = 9$,
the left member is $(-6) + 7 = 1$,
the right member is $(-8) + 9 = 1$.
Hence the truth set is $\{9\}$.

3. If $(-1) + 2 + (-3) = 4 + x + (-5)$ is true for some x,
then $(-2) = x + (-1)$ is true for the same x,
$(-2) + 1 = x + (-1) + 1$ is true for the same x,
$-1 = x$ is true for the same x.
If $x = -1$,
the left member is $(-1) + 2 + (-3) = (-2)$,
the right member is $4 + (-1) + (-5) = (-2)$.
The solution is -1.

4. If $(x + 2) + x = (-3) + x$ is true for some x,
then $2x + 2 = (-3) + x$ is true for the same x,
$2x + 2 + (-x) + (-2)$
$= (-3) + x + (-x) + (-2)$ is true for the same x,
$x = -5$ is true for the same x.
If $x = -5$,
the left member is $\big((-5) + 2\big) + (-5) = (-3) + (-5)$
$= -8$
the right member is $(-3) + (-5) = -8$
Hence the truth set is $\{-5\}$.

5. If $(-2) + x + (-3) = x + (-\frac{5}{2})$ is true for some x,
then $x + (-5) = x + (-\frac{5}{2})$ is true for the same x,
$(-x) + x + (-5)$
$= (-x) + x + (-\frac{5}{2})$ is true for the same x,
$-5 = -\frac{5}{2}$ is true for the same x;
but $-5 = -\frac{5}{2}$ is false, which contradicts
the assumption that the equation was true for
some x.
Hence the truth set is $\emptyset$.

[page 135]

6. If $|x| + (-3) = |-2| + 5$ is true for some x,
then $|x| + (-3) + 3 = 2 + 5 + 3$ is true for the same x,
$|x| = 10$ is true for the same x,
$x = 10$ or $x = -10$ is true for the same x.

If $x = 10$,
the left member is $|10| + (-3) = 10 + (-3)$
$= 7$,
the right member is $|-2| + 5 = 2 + 5$
$= 7$.

If $x = -10$,
the left member is $|-10| + (-3) = 10 + (-3)$
$= 7$,
the right member is $|-2| + 5 = 2 + 5$
$= 7$.

The solutions are 10, -10.

7. If $(-\frac{3}{8}) + |x| = (-\frac{3}{4}) + (-1)$ is true for some x,
then $\frac{3}{8} + (-\frac{3}{8}) + |x| = \frac{3}{8} + (-\frac{3}{4}) + (-1)$ is true for the same x,
$|x| = \frac{3}{8} + (-\frac{6}{8}) + (-\frac{8}{8})$ is true for the same x,
$|x| = -\frac{11}{8}$ is true for the same x.

But $|x|$ is non-negative for every x, which contradicts the assumption that the equation is true for some x. There is no solution.

8. If $x + (-3) = |-4| + (-3)$ is true for some x,
then $x + (-3) + 3 = 4 + (-3) + 3$ is true for the same x,
$x = 4$ is true for the same x.

If $x = 4$,
the left member is $4 + (-3) = 1$,
the right member is $|-4| + (-3) = 4 + (-3)$,
$= 1$.

Hence the truth set is $\{4\}$.

[page 135]

9. If $(-\frac{4}{3}) + (x+\frac{1}{2}) = x + (x+\frac{1}{2})$ is true for some x,

then $x + (-\frac{8}{6}) + \frac{3}{6} = 2x + \frac{3}{6}$ is true for the same x,

$x + (-x) + (-\frac{3}{6}) + (-\frac{5}{6}) = 2x + (-x) + (-\frac{3}{6}) + \frac{3}{6}$ is true for the same x.

$$-\frac{8}{6} = x$$

$$-\frac{4}{3} = x$$

If $x = -\frac{4}{3}$,

the left member is $(-\frac{4}{3}) + (-\frac{4}{3}+\frac{1}{2}) = (-\frac{8}{6}) + (-\frac{8}{6}) + \frac{3}{6}$

$= -\frac{13}{6}$,

the right member is $-\frac{4}{3} + (-\frac{4}{3}+\frac{1}{2}) = -\frac{8}{6} + (-\frac{8}{6}) + \frac{3}{6}$

$= -\frac{13}{6}$.

The solution is $-\frac{4}{3}$.

Pages 135-139. This is probably the student's first experience at anything approaching a formal proof. His chief difficulty here is to see the need for such a proof. We ask the student to extract from his experience the fact that for every number *there is* another number such that their sum is zero. At the same time the student can equally well extract from his experience that there is *only one* such number. Why then, do we accept the first idea from experience but prove the second? The reason is that we *can* prove the second. The two ideas differ in that one *must* be extracted from experience while the other *need not* be. The existence of the additive inverse is in this sense a more basic idea than the idea that there is only one such number. Speaking more formally, the *existence* of the additive inverse is an assumption; the *uniqueness* of the additive inverse is a theorem. You are referred to Haag, *Studies in Mathematics, Volume III, Structure of Elementary Algebra*, Chapter 2, Section 3, pp. 3.2-2.7, for further reading.

At this point we are still quite informal about proofs and try to lead into this kind of thinking gradually and carefully. The viewpoint about proofs in this course is not that we are trying to prove rigorously everything we say - we cannot at this stage - but that we are trying to give the students a little experience, within their ability, with the kind of thinking we call "proof". Don't frighten them by making a big issue of it, and don't be discouraged if some students do not immediately get the point. Discuss the proofs with them as clearly and simply as you can. We hope that by the end of the year they will have some feeling for deductive reasoning, a better idea of the nature of mathematics, and perhaps a greater interest in algebra because of the bearing of proof on the structure. For background reading on proofs the teacher is referred to Haag, Studies in Mathematics, Volume III, Structure of Elementary Algebra, Chapter 2, Section 3.

Page 135. (-3) added to 3 is 0.
4 added to -4 is 0.

Page 136. We added (-3) so that the addition property for opposites could be used to write the left member of the sentence in simpler form.

3, 5, -6.3 each have but one additive inverse. Since the set of real numbers is infinite, we cannot check all of them individually to confirm that the additive inverse of each is unique.

Page 137. We have used the addition property of equality and the addition property of zero. The next two reasons are the addition property of opposites and the addition property of zero.

Answers to Problem Set 6-5a; page 138.

1. In each part of this problem, the number for which the sentence is true is determined almost immediately by the uniqueness of the additive inverse; i.e. if $x + z = 0$, then $z = -x$.

(a) -3

(b) 2

(c) -8

(d) $\frac{1}{2}$

(e) -3

(f) $-\frac{2}{3}$

(g) $\frac{7}{3}$

(h) 3

(i) 3

2. Yes, for by Theorem 6-5a there is but one possible value for the variable, the opposite of the number to which the variable is added to make 0.

Page 139. Proof of Theorem 6-5b.

$(a+b)+\big((-a)+(-b)\big)$	$= a+b+(-a)+(-b)$	associative property of addition
	$= \big(a+(-a)\big)+\big(b+(-b)\big)$	commutative and associative properties of addition
	$= 0+0$	addition property of opposites
	$= 0$	addition property of zero
Thus $-(a+b) = (-a)+(-b)$		$(a+b)$ has only one additive inverse

Answers to Problem Set 6-5b; pages 139-140:

1. (a) $-(x+y) = (-x)+(-y)$ True; proved in Section 6-5.
   (b) $-x = -(-x)$ Since $-(-x) = x$, this is false for all real x except zero.
   (c) $-(-x) = x$ True for all real x.
   (d) $-\big(x+(-2)\big) = (-x)+2$ True; a special case of (a) where $y = -2$, $-y = 2$.
   (e) $-\big(a+(-b)\big) = (-a)+b$ True; a special case of (a) where $x = a$, $y = -b$, $(-x) = (-a)$, $(-y) = b$.
   (f) For $a = 2$, $b = 4$, the sentence becomes

$$\big(2+(-4)\big)+(-2) = 4$$
$$-4 = 4$$

which is false. The sentence is not true for all real numbers.

While this proof by counter example is sufficient, one may also reason as follows: Application of the associative and commutative properties of addition, the addition property of opposites, and the addition property of 0 leads to $-b = b$; in other words, If "$\big(a+(-b)\big)+(-a) = b$" is true for all real numbers a and b, then "$-b = b$" is true for all real numbers a and b. But "$-b = b$" is true only for $b = 0$. Therefore, the statement "$\big(a + (-b)\big) + (-a) = b$" is not true for all real numbers a and b.

(g) $-\big(x + (-x)\big) = x + (-x)$. True, a special case of (a) where $x = x$, $y = (-x)$, $(-y) = x$.

2.
$$\begin{aligned} (-x) + \big(y + (-z)\big) &= (-x) + \big((-z) + y\big) && \text{commutative property of addition.} \\ &= \big((-x) + (-z)\big) + y && \text{associative property of addition.} \\ &= \big(-(x+z)\big) + y && -(a+b) = (-a) + (-b). \\ &= y + \big(-(x+z)\big) && \text{commutative property of addition.} \end{aligned}$$

3. $-\big(3 + 6 + (-4) + 5\big) = (-3) + (-6) + 4 + (-5)$ is true. The opposite of the sum of any number of numbers is the sum of their opposites.

(a) True

(b) False

(c) True

(d) False

*4. To show that $-(a+b+c) = (-a) + (-b) + (-c)$, we need to show that

$$(a+b+c) + \big((-a) + (-b) + (-c)\big) = 0,$$

because one number is the opposite of another if their sum is zero.

Proof:

$$\begin{aligned} (a+b+c)+\big((-a)+(-b)+(-c)\big) &= \big(a+(-a)\big)+\big(b+(-b)\big)+\big(c+(-c)\big) && \text{associative and commutative properties of addition.} \\ &= 0 + 0 + 0 && \text{addition property of opposites.} \\ &= 0 && \text{addition property of 0.} \end{aligned}$$

Hence $-(a+b+c) = (-a)+(-b)+(-c)$.

This could be reasoned similarly for any number of numbers.

*5. For any real number a and any real number b and any real number c,

If	$a + c = b + c,$	
then	$(a + c) + (-c) = (b + c) + (-c)$	addition property of equality.
	$a + \left(c + (-c)\right) = b + \left(c + (-c)\right)$	associative property of addition.
	$a + 0 = b + 0$	addition property of opposites.
	$a = b$	addition property of 0.

Answers to Review Problems; pages 142-144:

1. (a) $3\left(8 + (-6)\right) = 3(2)$
$= 6$

(b) $-3 + 2 \times 3 = -3 + 6$
$= 3$

(c) $2 \times 7 + (-14) = 14 + (-14)$
$= 0$

(d) $(-\frac{2}{5}) + \frac{3}{5} = \frac{1}{5}$

(e) $|-6| \cdot |3| + (-3) = (18) + (-3)$
$= 15$

(f) $6\,(1 + |-4|) = 6(1 + 4)$
$= 6(5)$
$= 30$

2. (a) true
(b) true
(c) false
(d) true
(e) true
(f) true
(g) true

3. (a) The left numeral is

$\frac{2}{3} + \left(7 + (-\frac{2}{3})\right) = \frac{2}{3} + \left((-\frac{2}{3}) + 7\right)$	commutative property for addition.
$= \left(\frac{2}{3} + (-\frac{2}{3})\right) + 7$	associative property for addition.
$= 0 + 7$	addition property of opposites.
$= 7.$	addition property of zero.

The right numeral is 7.
Hence the sentence is true.

(b) The left numeral is

$|-5| + (-.36) + |-.36| = 5+\Big((-.36) + |.36|\Big)$ associative property for addition.

$= 5+\Big((-.36) + (.36)\Big)$ definition of absolute value.

$= 5+0$ addition property of opposites.

$= 5.$ addition property of zero.

The right numeral is $10+\Big((2+(-7)\Big) = 10+(-5)$

$= 5.$

Hence the sentence is true.

4. (a) If $\frac{5}{9} + 32 = x+\frac{5}{9}$ is true for some x,

then $\frac{5}{9} + 32+(-\frac{5}{9}) = x+\frac{5}{9}+(-\frac{5}{9})$ is true for the same x,

$\Big(\frac{5}{9} +(-\frac{5}{9})\Big) + 32 = x + 0$ is true for the same x,

$0 + 32 = x$ is true for the same x,

$32 = x$ is tuue for the same x.

If $x = 32$,

the left member is $\frac{5}{9} + 32 = 32\frac{5}{9}$,

the right member is $32 + \frac{5}{9} = 32\frac{5}{9}$.

Hence the truth set is $\{32\}$.

(b) If $x + 5 + (-x) = 12+(-x)+(-3)$ is true for some x,

then $x+5+(-x)+x = 9+(-x)+x$ is true for the same x,

$x + 5 = 9$ is true for the same x,

$x+5+(-5) = 9+(-5)$ is true for the same x,

$x = 4$ is true for the same x.

If $x = 4$,

the left member is $4+5+(-4) = 5$,

the right member is $12+(-4)+(-3) = 5$.

Hence the truth set is $\{4\}$.

(c) If $x + \frac{15}{2} + x = 10 + x + (-\frac{7}{2})$ is true for some x,

then $x + \frac{15}{2} + x + (-x) = 10 + x + (-\frac{7}{2}) + (-x)$ is true for the same x,

$\frac{15}{2} + x = 10 + (-\frac{7}{2})$ is true for the same x,

$\frac{15}{2} + x + (-\frac{15}{2}) = 10 + (-\frac{7}{2}) + (-\frac{15}{2})$ is true for the same x,

$x = 10 - 11$ is true for the same x,

$x = -1$ is true for the same x.

If $x = -1$,

the left member is $(-1) + \frac{15}{2} + (-1) = (-2) + \frac{15}{2}$

$= \frac{11}{2}$

the right member is $10 + (-1) + (-\frac{7}{2}) = 9 + (-\frac{7}{2})$

$= \frac{11}{2}$

Hence the truth set is $\{-1\}$.

(d) If $|x| + 3 = 5 + |x|$ is true for some x,

then $|x| + 3 + (-|x|) = 5 + |x| + (-|x|)$ is true for the same x,

$3 = 5$ is true for the same x.

But $3 = 5$ is false, which contradicts the assumption that the equation was true for some x. Hence the truth set is $\emptyset$.

5. (a) $|3| + |a| > |-3|$ is true for the set of all numbers except 0.
   (b) $|3| + |a| = |-3|$ is true for $\{0\}$.
   (c) $|3| + |a| < |-3|$ is true for $\emptyset$.

[page 142]

6. (a) Either both are negative; or one is negative and the other is either positive or 0, and the negative number has the greater absolute value.
   (b) One is the opposite of the other.
   (c) Either both are positive; or one is positive and the other is either negative or 0, and the positive number has the greater absolute value.

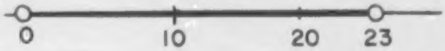
7. If x is the number of dollars in the week's sales,

$$80 + .03x = 116$$

8. (a) If x is the length of the fourth side,

$$x > 0 \text{ and } x < 23$$

   (b)



9. (a) $2ab + ac$
   (b) $2ab + 2ac$
   (c) $3(a+b)$
   (d) $5x(1+2a)$
   (e) $xy(x+1)$
   (f) $2ab(3a+b)$
   (g) $a^2bc + 3ab^2$
   (h) $3a^2 + 6ab + 9ac$

10. (a) Yes, the set is closed under the operation of "opposite".
    (b) Yes, the set is closed under the operation of "absolute value".
    (c) Yes, if a set is closed under "opposite" it is closed under "absolute value", since either the number or its opposite is the absolute value of the number.

11. (a) Yes, the set is closed under the operation of "absolute value".
    (b) No, the set is not closed under the operation of "opposite".
    (c) No, if a set is closed under "absolute value", it is not necessarily closed under "opposite", since even if the absolute value of a number is in the set, the opposite of the number may not be.

12. Since in each hour the faster car covers 10 miles more than the other, they will be m miles apart in $\frac{m}{10}$ hours.

CHAPTER 6

Suggested Test Items

1. Find the common names for the following:

   (a) $(-7) + (-17)$
   (b) $(-13) + 49 + (-24)$
   (c) $(-3) + |(-3)| + |(-5)|$
   (d) $(-3) + |(-3) + (-5)|$
   (e) $(-47) + 18(0)$
   (f) $n + (-n)$

2. Write a common numeral for each of the following numbers, without using paper for calculations. In each case tell what properties you used to make your work easier.

   (a) $(287) + (-287)$
   (b) $(-17) + 30 + (-83)$
   (c) $19 + 954 + (-19)$

3. The following sentences are true for every a, every b, and every c:

   A. $a + b = b + a$
   B. $a + (-a) = 0$
   C. $(a + b) + c = a + (b + c)$
   D. If $a = b$, then $a + c = b + c$
   E. $-(a+b) = (-a) + (-b)$
   F. $-(-a) = a$
   G. $a + 0 = a$

   Which of the sentences expresses:

   (a) the commutative property of multiplication
   (b) the addition property of zero
   (c) the addition property of equality
   (d) the fact that the opposite of the sum of two numbers is the sum of the opposites

4. If m and n are negative numbers, which of the following are true sentences:

   (a) $m + n = -(|m| + |n|)$

   (b) $|m| = |n|$

(c) $m + n < 0$
(d) $|m| + |n| > 0$
(e) $(-m) + (-n) \not< 0$

5. Find the truth set of each of the following open sentences:

   (a) $x + 2 = 7$
   (b) $0 = 7 + n$
   (c) $m + (-6) = 0$
   (d) $(-6) + 7 = (-8) + a$
   (e) $|x| + (-2) = 1$
   (f) $(-3) + w = (-4) + (-3)$
   (g) $|x + 3| = 7$

6. Given the set $K = \{0, \frac{1}{2}, -\frac{1}{2}, -3, 3\}$

   (a) Is the set K closed under the operation of taking the opposite of each element?
   (b) Write the set S of all sums of pairs of elements of set K.
   (c) Is set S a subset of set K? Is set K closed under addition of pairs of elements?

7. When a certain number is added to 99 the result is 287.

   (a) Write an open sentence to find the number.
   (b) Find the number by finding the truth set of the sentence.

8. A mother is 28 years older than her son. The mother's age is equal to the son's age plus 10 years more than the son's age.

   (a) Write an open sentence to find the son's age. (Hint: Find two phrases for the mother's age. Do they name the same number?)
   (b) Find the truth set of this sentence.

9. Jim said "When I buy three dozen more marbles I'll have three more than 13 dozen."

   (a) Write an open sentence from which you can find out how many marbles Jim has.

(b) Find the truth set of this sentence.

(c) Show how the use of the distributive property helps make the calculation of this truth set easier. (Hint: Write the open sentence leaving the numbers of dozens as indicated products.)

Chapter 7

PROPERTIES OF MULTIPLICATION

This is the second of three chapters in which the operations with the numbers of arithmetic are extended to the real numbers and the properties of these operations are brought out. You may want to refer back to the statement at the beginning of the commentary for Chapter 6 for a more detailed statement about these three chapters.

Background reading for the mathematics of this chapter is available in Studies in Mathematics, III, Chapter 2, Section 3 and Chapter 3, Section 4.

7-1. Multiplication of Real Numbers

There are several ways of making multiplication of real numbers seem plausible. It seems best to let the choice of definition of multiplication be a necessary outgrowth of a desire to retain the distributive property for real numbers.

Page 145. The properties listed are the commutative and associative properties of multiplication, the multiplication properties of 1 and 0, and the distributive property.

Examples are given of all possible cases of multiplication of pairs of positive and negative numbers and zero.

Page 146. Make clear to the students that what we are doing here is not a proof. We couldn't prove anything about something which has not been defined. However, to guide us in choosing the definition we ask "If we had a definition of the product ab for negative numbers, how would the numbers behave under the distributive property?" We find that they would behave in such a way that

$$0 = 6 + (2)(-3)$$

would have to be true. But if the uniqueness of the additive inverse is to continue to hold, $(2)(-3)$ would then have to be the opposite of 6.

Page 146. $|3|\,|2| = 6$ and $|-2|\,|-3| = 6$. These name the same number as $(3)(2)$ and $(-2)(-3)$. $(-3)(4)$ is the same number as $-(|-3|\,|4|)$; $(-5)(-3) = |-5|\,|-3|$; and $(0)(-2)$ is $|0|\,|-2|$.

Page 147. As in the case of addition, the point of view here is that we extend the operation of multiplication from the numbers of arithmetic to all real numbers so as to preserve the fundamental properties. This actually forces us to define multiplication in the way we have. In other words it could not be done in any other way without giving up some of the properties.

The general definition of multiplication for real numbers is stated in terms of absolute values because $|a|$ and $|b|$ are numbers of arithmetic, and we already know how to multiply numbers of arithmetic. The only problem for real numbers is to determine whether the product is positive or negative.

There are a number of devices available to help the teacher who feels a need to make the definition of multiplication plausible to his students. Below are two forms of one scheme of this sort, a scheme which leads the student to the same results as the definition of multiplication by asking him to recognize a pattern in the sequence of products. In using either form of this scheme, however, the teacher should bear in mind that this is a weaker approach than that of the text; for while the definition in the text is based on considerations of mathematical structure, the device below has only the appeal of an implied extension of the symmetry of a multiplication table.

(In either form below the student fills in the missing products.)

(1)	$(3)(2) = 6$	$(3)(-2) = -6$
	$(2)(2) = 4$	$(2)(-2) = -4$
	$(1)(2) = 2$	$(1)(-2) = -2$
	$(0)(2) = 0$	$(0)(-2) = 0$
	$(-1)(2) =$	$(-1)(-2) =$
	$(-2)(2) =$	$(-2)(-2) =$

(2)

x	3	2	1	0	-1	-2	-3
3	9	6	3	0			
2	6	4	2	0			
1	3	2	1	0			
0	0	0	0	0			
-1							
-2							
-3							

Answers to Problem Set 7-1; pages 148-150:

1. (a) $(-7)(-8) = |-7|\ |-8| = 56$

   (b) $(\frac{2}{3})(-12) = -(|\frac{2}{3}|\ |-12|) = -\left((\frac{2}{3})(12)\right) = -8$

   (c) $$\begin{aligned} |(-3)(2)|\ (-2) &= \left|-(|-3|\ |2|)\right|\ (-2) \\ &= |-6|\ (-2) \\ &= (6)(-2) \\ &= -(|6|\ |-2|) \\ &= -12 \end{aligned}$$

   (d) $$\begin{aligned} (-18)(\tfrac{3}{5}) &= -(|-18|\ |\tfrac{3}{5}|) \\ &= -\left((18)(\tfrac{3}{5})\right) \\ &= -\frac{54}{5} \end{aligned}$$

   (e) $(-\frac{3}{4})(-\frac{2}{5}) = |-\frac{3}{4}|\ |-\frac{2}{5}| = \frac{3}{10}$

   (f) $$\begin{aligned} |-2|\ \left((-3) + |-3|\right) &= 2\left((-3) + 3\right) \\ &= 2(0) = 0 \end{aligned}$$

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2. (a) $(-\frac{1}{2})(-4) = 2$

(b) $\left((-\frac{1}{2})(2)\right)(-5) = (-1)(-5) = 5$

(c) $(-\frac{1}{2})\left((2)(-5)\right) = (-\frac{1}{2})(-10) = 5$

(d) $(-3)(-4) + (-3)(7) = 12 + (-21) = -9$

(e) $(-3)\left((-4) + 7\right) = (-3)(3) = -9$

(f) $(-3)(-4) + 7 = 12 + 7 = 19$

(g) $|-3|(-4) + 7 = 3(-4) + 7 = (-12) + 7 = -5$

(h) $|-3|\,|-2| + (-6) = (3)(2) + (-6) = 6 + (-6) = 0$

(i) $(-3)\,|-2| + (-6) = (-3)(2) + (-6) = (-6) + (-6) = -12$

(j) $(-3)\left(|-2| + (-6)\right) = (-3)\left(2 + (-6)\right) = (-3)(-4) = +12$

(k) $(-0.5)\left(|-1.5| + (-4.2)\right) = (-0.5)(-2.7) = 1.35$

3. (a) $2(-2) + 7(3) = (-4) + 21 = 17$

(b) $3\left(-(-2)\right) + \left((-4)(3) + 7\left(-(-4)\right)\right) = 6 + \left((-12) + 28\right)$

$= 6 + 16 = 22$

(c) $(-2)^2 + 2\left((-2)(-4)\right) + (-4)^2 = 4 + 2(8) + 16$

$= 4 + 16 + 16 = 36$

(d) $\left((-2) + (-4)\right)^2 = (-6)^2 = 36$

(e) $(-2)^2 + \left(3|-4| + (-4)(3)\right) = 4 + \left(12 + (-12)\right) = 4 + 0 = 4$

(f) $|(-2) + 2| + (-5)|(-3) + (-4)| = 0 + (-5)(7) = -35$

4. (a) $2(-10) + 8 = 12$ false

(b) $2\left(-(-10)\right) + 8 = 28$ true

(c) $(-3)\left((2)(-2)\right) + 8 \neq 20$

$(-3)(-4) + 8 \neq 20$ false

(d) $(-5)\left((-2)(-4) + 30\right) < 0$

$(-5)(8 + 30) < 0$ true

(e) $|2 + 3| + (-2)\Big(|2 + (-4)|\Big) \geq 1$

$5 + (-2)(|-2|) \geq 1$

$5 + (-4) \geq 1$ true

5. (a) If $x + (-3)(-4) = 8$ is true for some x,

then $x + 12 = 8$ is true for the same x,

$x + 12 + (-12) = 8 + (-12)$ is true for the same x,

$x = -4$ is true for the same x.

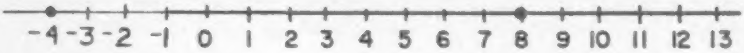
If $x = -4$,

The left member is $(-4) + (-3)(-4) = (-4) + (12)$

$= 8$

The right member is 8.

Hence the truth set is $\{-4\}$.



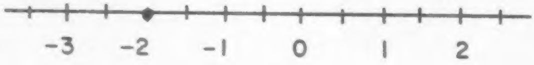
The form suggested above should be used as long as you think it is helpful to your students but not longer. When, in your judgment, writing "is true for some x" and "is true for the same x" has served its purpose, you should no longer require it.

The important thing is to make sure that the students do not lose sight of the ideas they are using here. Find opportunities from time to time to help the students keep in mind why a number which makes the first sentence true will necessarily make the last sentence true.

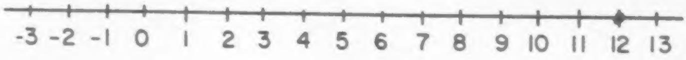
Some teachers may wonder why we do not check the solution by "substituting" -4 for x in the sentence. In fact, we have not used the word "substitute" in this context for the following reason. The variable x is a symbol representing an unspecified number of a given set. We say that the "value of x is 3" when we specify that x represents 3. If we were to say "substitute 3 for x", the student might get the impression that the symbol x is somehow erased and replaced by the symbol 3. We want the student to understand that in such a case x is 3, not replaced by 3.

The truth sets and graphs for the remaining parts of the problem are:

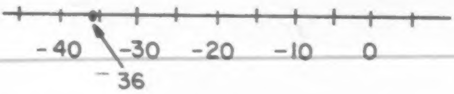
(b) $\{-2\}$



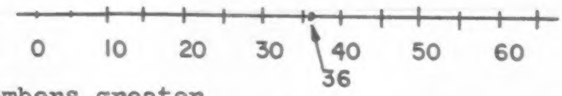
(c) $\{12\}$


-3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11 12 13

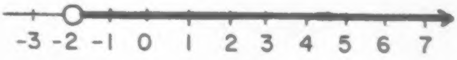
(d) $\{-36\}$


-40
-30
-20
-10
0
-36

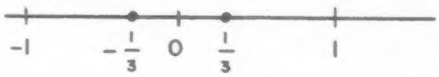
(e) $\{36\}$


0
10
20
30
40
50
60
36

(f) all numbers greater than -2


-3 -2 -1 0 1 2 3 4 5 6 7

(g) $\{\frac{1}{3}, -\frac{1}{3}\}$


-1
$-\frac{1}{3}$
0
$\frac{1}{3}$
1

6. $P = \{-12, -8, -3, -2, 1, 4, 6, 9, 16\}$ Making a table of products may facilitate the student's work in this problem.

7. Q is the set of all real numbers. The set of real numbers is closed under multiplication.

8. T is the set of all positive real numbers. The set of negative real numbers is not closed under multiplication.

9. Referring to Problem 6, above, we see that the subset of P made up of all positive numbers in P is $K = \{1, 4, 6, 9, 16\}$.

*10. If $a = 0$ or $b = 0$, then $ab = 0$, $|ab| = 0$, and $|a| = 0$ or $|b| = 0$, so $|ab| = |a| \cdot |b|$.

If $a \neq 0$ and $b \neq 0$, exactly one of the following is true:

$ab = |a| \cdot |b|$	Definition of the product
or	of two real numbers.
$ab = -(|a| \cdot |b|)$	

(Note that the two possible values of ab are opposites.)

[page 149]

Then $|ab|$ is either $|a| \cdot |b|$ or $-(|a| \cdot |b|)$, whichever is greater	Definition of absolute value.
But $|a| \cdot |b|$ is positive	Definition of the product of two real numbers.
and $-(|a| \cdot |b|)$ is negative	Definition of opposites.
Hence $|ab| = |a| \cdot |b|$	A positive number is greater than a negative number.

11. (a) Either both are positive or both are negative.
    (b) One is positive and one is negative.
    (c) b is positive.
    (d) b is negative.
    (e) b is negative.
    (f) b is positive.

*12. 1. Comparison property.
    2. Multiplication property of 0.
    3. Definition of multiplication.
    4. Definition of opposites - if a number is positive, its opposite is negative.

This is an example of a proof by contradiction. In spite of this the proof is straightforward, and can be followed without a discussion of formal proof by contradiction, which is dealt with in Section 7-8 of this chapter.

7-2. Properties of Multiplication

Again we should emphasize that the demand for the familiar properties to be satisfied suggested to us what the definition of multiplication must be. On the other hand, we could have given the definition of multiplication at the outset without any motivation whatsoever. Thus the definition itself does not actually involve an assumption of the properties. It is even conceivable that, in spite of being more or less forced to define multiplication in this way, we might find that the properties still do not hold. Hence the need remains to prove that multiplication so defined does really have the properties. In doing this we have *not* fallen into the logical trap of arguing on a circle.

Pages 150-152. You may want to mention the closure property of multiplication at this point.

The property of 1, which is stated in the form $a \cdot 1 = a$, could just as well have been stated in the form $1 \cdot a = a$. Some like to combine these in the statement $a \cdot 1 = 1 \cdot a = a$. There is no objection to your doing this.

Reasons for the multiplication property of 1 are as follows:

If $a < 0$, $a \cdot 1 = -(\lvert a \rvert \cdot 1)$	definition of multiplication of real numbers.
$= -\lvert a \rvert$	multiplication property of 1 for $a > 0$.
$= a$	definition of absolute value, and the opposite of the opposite of a number is the number itself.

A proof for the multiplication property of 0 follows:

For any real number a,

if $a \geq 0$, $a \cdot 0 = \lvert a \rvert \cdot \lvert 0 \rvert$	definition of multiplication of real numbers.
But if $a \geq 0$, $\lvert a \rvert = a$	definition of absolute value.
$\lvert 0 \rvert = 0$	

Thus $a \cdot 0$ is the product of two numbers of arithmetic,

and $a \cdot 0 = 0$.

If $a < 0$, $a \cdot 0 = -(\lvert a \rvert \cdot \lvert 0 \rvert)$	definition of multiplication of real numbers.
$= -(0)$	product of two numbers of arithmetic
$= 0$	the opposite of 0 is 0.

Answers to Problem Set 7-2a; page 152:

1. (a) $\left((-\frac{3}{4})(-\frac{4}{3})\right)(-17) = (1)(-17) = -17$

(b) $(-8)\left(5 + (-6)(\frac{2}{3})\right) = (-8)\left(5 + (-4)\right) = (-8)(1) = -8$

(c) $(-4)\left((-\frac{3}{2})(4) + (-\frac{6}{5})(-5)\right) = (-4)\left((-6) + 6\right) = (-4)(0) = 0$

(d) $\left((4)(-6) + (-8)(-3)\right)(-\frac{47}{13}) = \left((-24) + (24)\right)(-\frac{47}{3}) = (0)(-\frac{47}{3}) = 0$

2. (a) Done in text.

(b) $(3)(-5) = -(|3|\ |-5|) = -15$

$(-5)(3) = -(|-5|\ |3|) = -15$

(c) $(-\frac{2}{3})(0) = -(|-\frac{2}{3}|\ |0|) = 0$

$(0)(-\frac{2}{3}) = -(|0|\ |-\frac{2}{3}|) = 0$

(d) $(3)(-4) = |-3|\ |-4| = 12$

$(-4)(-3) = |-4|\ |-3| = 12$

(e) $(-7)(\frac{5}{7}) = -(|-7|\ |\frac{5}{7}|) = -5$

$(\frac{5}{7})(-7) = -(|\frac{5}{7}|\ |-7|) = -5$

Pages 153-154. $|(ab)c| = |ab|\ |c|$ It has been stated in the text and proven in Problem *10 of Problem Set 7-1 that $|ab| = |a|\ |b|$. Note that this property also justifies the next step.

If one of a, b, c is 0, then $(ab)c = 0$ and $a(bc) = 0$ by the multiplication property of 0.

The proof for associativity is probably too much to demand of all students. However, the idea is not difficult and might appeal to the better students.

Answers to Problem Set 7-2b; page 154:

1.

If a is	+	+	+	+	-	-	-	-
and b is	+	+	-	-	+	+	-	-
and c is	+	-	+	-	+	-	+	-
then ab is	+	+	-	-	-	-	+	+
bc is	+	-	-	+	+	-	-	+
(ab)c is	+	-	-	+	-	+	+	-
a(bc) is	+	-	-	+	-	+	+	-

In the text we have seen that for all real numbers a, b, and c,

$$|(ab)c| = |a(bc)|$$

and that if one of a, b, c is 0, the associative property for multiplication holds.

If $$|(ab)c| = |a(bc)|$$

exactly one of

$$(ab)c = a(bc)$$

or

$$(ab)c = -a(bc)$$

is true; that is, either the numerals must name the same number, or the numbers are opposites. Since the table reveals that for all possible combinations of positive and negative values for a, b, and c, (ab)c and a(bc) are either both positive or both negative, it is clear that (ab)c and a(bc) are not opposites, and that (ab)c = a(bc).

2. (a) $\big((3)(2)\big)(-4) = (6)(-4) = -24$

$(3)\big((2)(-4)\big) = (3)(-8) = -24$

(b) $\big((3)(-2)\big)(-4) = (-6)(-4) = 24$

$(3)\big((-2)(-4)\big) = (3)(8) = 24$

(c) $\big((3)(-2)\big)(4) = (-6)(4) = -24$

$(3)\big((-2)(4)\big) = (3)(-8) = -24$

(d) $\big((-3)(2)\big)\,(-4) = (-6)(-4) = 24$

$(-3)\,\big((2)(-4)\big) = (-3)(-8) = 24$

(e) $\big((-3)(-2)\big)\,(-4) = (6)(-4) = -24$

$(-3)\,\big((-2)(-4)\big) = (-3)(8) = -24$

(f) $\big((-3)(-2)\big)\,(0) = (-6)(0) = 0$

$(3)\,\big((-2)(0)\big) = (3)(0) = 0$

3. (a) $(-5)(17)(-20)(3) = \big((-5)(-20)\big)\,\big((17)(3)\big)$

$= (100)(51)$

$= 5100$

(b) $(\frac{2}{3})(\frac{7}{5})(-\frac{3}{4}) = \big((\frac{2}{3})(-\frac{3}{4})\big)\,(\frac{7}{5})$

$= (-\frac{1}{2})(\frac{7}{5})$

$= -\frac{7}{10}$

(c) $(\frac{1}{3})(\frac{6}{5})(-21) = \big((\frac{1}{3})(-21)\big)\,(\frac{6}{5})$

$= (-7)(\frac{6}{5})$

$= -\frac{42}{5}$

(d) $(-\frac{2}{3})(\frac{4}{5})(\frac{3}{2})(-\frac{5}{4}) = \big((-\frac{2}{3})(\frac{3}{2})\big)\,\big((\frac{4}{5})(-\frac{5}{4})\big)$

$= (-1)(-1)$

$= 1$

(e) $(\frac{1}{5})(-19)(-3)(50) = \big((\frac{1}{5})(50)\big)\,\big((-19)(-3)\big)$

$= (10)(57)$

$= 570$

(f) $(-7)(-25)(3)(-4) = \big((-7)(3)\big)\,\big((-25)(-4)\big)$

$= (-21)(100)$

$= -2100$

Page 155.

$(5)\big(2 + (-3)\big)$ and $(5)(2) + 5(-3)$ are numerals for -5.

$(5)\big((-2) + (-3)\big)$ and $(5)(-2) + (5)(-3)$ are numerals for -25.

and $(-5)\big((-2) + (-3)\big)$ and $(-5)(-2) + (-5)(-3)$ are numerals for 25.

The following discussion makes it evident why we do not suggest that the student, even the eager one, be encouraged to try to prove the distributive property.

A complete verification of the distributive law requires an examination of twelve cases for $a(b + c)$:

(1) $a \geq 0$, $b \geq 0$, $c \geq 0$;
(2) $a < 0$, $b < 0$, $c < 0$;
(3) $a \geq 0$, $b \geq 0$, $c < 0$, $b \geq |c|$;
(4) $a \geq 0$, $b \geq 0$, $c < 0$, $b < |c|$;
(5) $a \geq 0$, $b < 0$, $c \geq 0$, $|b| \geq c$;
(6) $a \geq 0$, $b < 0$, $c \geq 0$, $|b| < c$;
(7) $a < 0$, $b \geq 0$, $c \geq 0$;
(8) $a \geq 0$, $b < 0$, $c < 0$;
(9) $a < 0$, $b \geq 0$, $c < 0$, $b \geq |c|$;
(10) $a < 0$, $b \geq 0$, $c < 0$, $b < |c|$;
(11) $a < 0$, $b < 0$, $c \geq 0$, $|b| \geq c$;
(12) $a < 0$, $b < 0$, $c \geq 0$, $|b| < c$.

There are eight further cases to be considered for $ab + ac$.

The student should be deterred from trying to prove this distributive property. As in the verification of the associative law for addition, the number of cases involved is one stumbling block. Here, though, there is a further difficulty: The proof requires that the student *distribute multiplication over subtraction* for numbers of arithmetic. In case (3), for example,

$a(b + c)$ is, by definition, $a(b-|c|)$, and we need to know that

$$a(b - |c|) = ab - a|c|$$

before we can proceed. This equality holds because of the distributive property mentioned. It is implicit in this property that $ab > a|c|$. The definitions of addition and multiplication then allow us to write

$$\begin{aligned} a(b - |c|) &= ab - a|c| \\ &= ab + (-a|c|) \\ &= ab + ac. \end{aligned}$$

We have barely mentioned subtraction for the numbers of arithmetic, so we could hardly expect the student to be aware of the distributive property of multiplication over subtraction for such numbers.

Answers to Problem Set 7-2c; page 155:

1. $(-9)(-92) + (-9)(-8) = -9 \big((-92) + (-8)\big) = -9\ (-100) = 900$
2. $(.63)(6) + (-1.63)(6) = \big(.63 + (-1.63)\big)\ 6 = (-1)(6) = -6$
3. $-\frac{3}{2}\big((-4) + 6\big) = -\frac{3}{2}\ (2) = -3$
4. $(-7)(-\frac{3}{4}) + (-7)(\frac{1}{3}) = -7\ \big((-\frac{3}{4}) + (\frac{1}{3})\big) = -7\ (-\frac{5}{12}) = \frac{35}{12}$
5. $(-\frac{3}{4})\ \big((-93) + (-7)\big) = -\frac{3}{4}\ (-100) = 75$
6. $(-7)(\frac{2}{3}) + (-5)(\frac{2}{3}) = \big((-7) + (-5)\big)\ \frac{2}{3} = (-12)(\frac{2}{3}) = -8$

Page 155. Reasons for the indicated steps in the proof that $(-1)a = -a$ are:

$a + (-1)a = 1(a) + (-1)a$	multiplication property of 1.
$= \big(1 + (-1)\big)a$	distributive property.
$= (0)a$	addition property of opposites.
$= 0$	multiplication property of 0.

Answers to Problem Set 7-2d; page 156:

1. *Theorem:* For any real numbers a and b, $(-a)(b) = -(ab)$

$(-a)(b) = \big((-1)(a)\big)(b)$	$(-1)x = -x$
$= (-1)(ab)$	associative property for multiplication
$= -(ab)$	$(-1)x = -x$

2. *Theorem:* For any real numbers a and b, $(-a)(-b) = ab$

$(-a)(-b) = \big((-1)\cdot a\big)\big((-1)\cdot b\big)$	$(-1)x = -x$
$= \big((-1)(-1)\big)(ab)$	associative and commutative properties for multiplication.
$= (1)ab$	definition of multiplication.
$= ab$	multiplication property of 1.

3. (a) $(-5)(ab) = -5ab$

(b) $(-2a)(-5c) = 10ac$

(c) $(3x)(-7y) = -21xy$

(d) $(-5c)(\frac{3}{5}d) = -3cd$

(e) $(\frac{2}{9}bc)(-6a) = -\frac{4}{3}abc$

(f) $(-0.5d)(1.2c) = -0.6cd$

7-3 and 7-4. Use of the Multiplication Properties

These sections introduce some of the necessary techniques of algebra. We wish to give sufficient practice with these techniques, but we wish also to keep them closely associated with the ideas on which they depend. We have to walk a narrow path between, on the one hand, becoming entirely mechanical and losing sight of the ideas and, on the other hand, dwelling on the ideas to the extent that the student becomes slow and clumsy in the algebraic manipulation. A good slogan to follow here is that manipulation must be based on understanding. The student must *earn the right* to "push symbols" (skipping steps, computing without giving reasons, etc.) by first mastering the ideas which lie behind and give meaning to the manipulation of the symbols.

Answers to Problem Set 7-3a; page 157:

1. (a) $3x + 15$
   (b) $7a + (-ak)$
   (c) $2a + 2b + 2c$
   (d) $(-9a) + (-9b)$
   (e) $3p + (-3q)$
   (f) $(-y) + z + (-5)$
   (g) $13y + xy$
   (h) $32 + 8m$
   (i) $(-gr) + (-g) + gs + gt$

2. (e), (f), (h), (i)

3. (a) $5(a + b)$
   (b) $(-9)(b + c)$
   (c) $6 (2 + 3)$ or $6 (5)$
   (d) $3(x + y + z)$
   (e) $k(m + p)$
   (f) $(a + b)(x + y)$
   (g) $(7 + 3)(\frac{1}{8})$ or $10 (\frac{1}{8})$
   (h) $(-6) (a^2 + b^2)$
   (i) $c(a + b + 1)$
   (j) $2\big(a + (-b)\big)$

4. (a) $19t$
   (b) $-6a$
   (c) $9y$
   (d) $13z$
   (e) $(-11m)$
   (f) $2a$
   (g) $4.0b$
   (h) $8x$
   (i) $3a + 7y$
   (j) $16p$
   (k) 0
   (ℓ) $2a + 19b$

Page 158. In collecting terms we want the direct application of the distributive property to be the main thought. Don't give the impression that collecting terms is a new process. We are avoiding the phrases "like terms" and "similar terms" because they are unnecessary and tend to encourage manipulation without understanding.

Answers to Problem Set 7-3b; pages 158-159:

1. (a) $13x$
   (b) $-13a$
   (c) $9k$
   (d) $3b$
   (e) n
   (f) $9x$
   (g) $-14a$
   (h) $2a$
   (i) $17p$
   (j) 0
   (k) $12a + 3c + 3c^2$
   (ℓ) $6a + 4b + c$
   (m) $6p + 11q$
   (n) $3x^2 + (-x) + 1$

2. In parts (f) and (g) the multiplication property of one.
   In part (k) the associative property of addition.
   In part (m) the commutative and associative properties of addition.

3. Since we have not yet introduced the multiplication property of equality, the pupil will have to go back to "guessing" the truth set, after collecting terms and using the addition property of equality where possible.
   (a) $\{2\}$
   (b) $\{-4\}$
   (c) $\emptyset$
   (d) $\{-9\}$
   (e) $\emptyset$
   (f) the set of all real numbers
   (g) $\{7\}$
   (h) $\{11\}$
   (i) $\{1\}$
   (j) $\{2\}$

Page 159. The reasons for the steps of the example are:

$(3x^2y)(7ax) = 3\cdot x\cdot x\cdot y\cdot 7\cdot a\cdot x$	associative property of multiplication.
$= 3\cdot 7\cdot a\cdot x\cdot x\cdot x\cdot y$	commutative property of multiplication.
$= (3\cdot 7)\cdot a\cdot (x\cdot x\cdot x)\cdot y$	associative property of multiplication.
$= 21ax^3y$	multiplication.

Answers to Problem Set 7-4a; page 160:

1. $-24b$
2. $-12c^2$
3. $-72b$
4. $42yz$
5. $18bc^2$
6. $-15w^4$
7. $-12ay^3$
8. $\frac{3}{8}ab^2c^2d$
9. $48p^2q^2$
10. $200b^3c^2d$
11. $$\begin{aligned}(\tfrac{1}{3}ab)(9a^2) &= \tfrac{1}{3}\cdot a\cdot b\cdot 9\cdot a\cdot a\\ &= (\tfrac{1}{3}\cdot 9)(a\cdot a\cdot a)\cdot b\\ &= 3a^3b\end{aligned}$$
12. $28abc$
13. $24a^2x^2$
14. $12a^3b^2c$

Answers to Problem Set 7-4b; page 161:

1. $(-3c) + (-3d)$
2. $16 + (-6b) + 14b^2$
3. $18xy + 6xz$
4. $(-12b^3c^2) + (-21b^2c^3)$
5. $5x^2 + 30x$
6. $20b^3 + 70b^2 + (-40b)$
7. $(-p) + (-q) + (-r)$
8. $(-21a) + 35b$
9. $12x^2y + 18x^2y^2 + 24xy^2$
10. $(-a^2) + (-2ab) + (-b^2)$
11. $(-8ac) + 20bc + 4c^2$
12. $(-x^2) + x$

Answers to Problem Set 7-4c; pages 161-162:

1. Encourage the pupil to do each of these in the form indicated for (a)

   (a) $$\begin{aligned}(x + 8)(x + 2) &= (x + 8)x + (x + 8)2\\ &= x^2 + 8x + 2x + 16\\ &= x^2 + 10x + 16\end{aligned}$$

   (b) $y^2 + (-8y) + 15$

   (c) $6a^2 + (-17a) + 10$

   (d) $a^2 + 4a + 4$

   (e) $x^2 + (-36)$

   (f) $y^2 + (-9)$

2. For real numbers a, b, c, d,

$$\begin{aligned}(a + b)(c + d) &= (a + b)c + (a + b)d \\ &= ac + bc + ad + bd \\ &= ac + (bc + ad) + bd\end{aligned}$$

When the pupil discovers that this gives him a formula for multiplying expressions of the form $(a + b)(c + d)$ he may want to use it instead of the longer form indicated in Problem 1. He should be ehcouraged to do so as soon as he is ready to use it with understanding and accuracy.

3. (a) $a^2 + 4a + 3$ (d) $y^3 + (-6y^2) + 9y + (-4)$
   (b) $6x^2 + 17x + 12$ (e) $m^2 + 6m + 9$
   (c) $ab + cb + ad + cd$ (f) $14 + 9z + z^2$
4. (a) $3a^2 + 5a + 2$ (d) $6p^2q^2 + (-10pq) + (-56)$
   (b) $4x^2 + 23x + 15$ (e) $16 + (-14y) + y^2 + y^3$
   (c) $8 + 13n + 5n^2$ (f) $15y^2 + (-11xy) + (2x^2)$

7-5. Multiplicative Inverse (page 162)

A given real number remains unchanged when 0 is added, by the addition property of zero. A given real number remains unchanged when it is multiplied by 1, by the multiplication property of one.

If d is a multiplicative inverse of c, then $cd = 1$. By the commutative property of multiplication $dc = 1$ and, so, c is a multiplicative inverse of d.

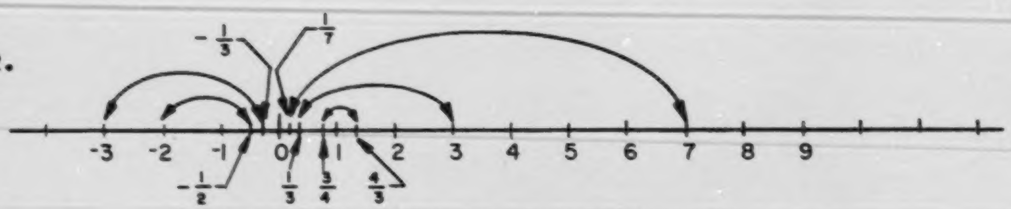
Pages 162-163. To test whether a pair of numbers are really multiplicative inverses, they can be multiplied, and if their product is one, they are multiplicative inverses.

When pairing numbers on the number line to form pairs of inverses, it is found that 0 cannot be paired with any number. As the student chooses positive numbers closer and closer to 0, he finds that the multiplicative inverses are larger and larger; as he chooses negative numbers greater than -1, he finds that the multiplicative inverses are further and further to the left on the number line. Thus there is no suggestion of a possible multiplicative inverse for 0 on the number line. Furthermore, the multiplication property of 0 shows that there is no number b such that $0 \times b = 1$. The student should conclude that there is no multiplicative inverse of 0. Formal proof of this follows in Section 7-8.

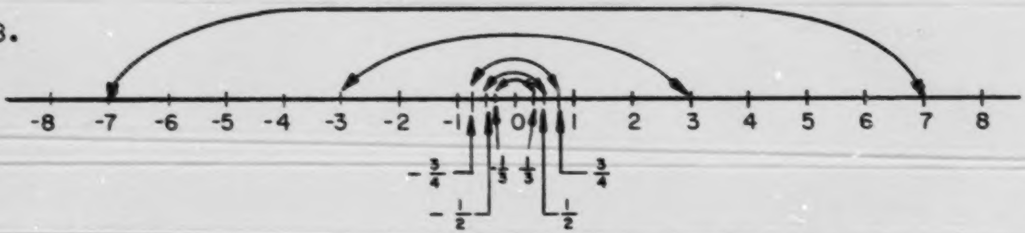
Answers to Problem Set 7-5; pages 164-165:

1. Number	Inverse under Multiplication	Number	Inverse under Multiplication
3	$\frac{1}{3}$	$-\frac{3}{7}$	$-\frac{7}{3}$
$\frac{1}{2}$	2	-7	$-\frac{1}{7}$
-3	$-\frac{1}{3}$	$\frac{3}{10}$	$\frac{10}{3}$
$-\frac{1}{2}$	-2	$\frac{1}{100}$	100
$\frac{3}{4}$	$\frac{4}{3}$	$-\frac{1}{100}$	-100
7	$\frac{1}{7}$	0.45	$\frac{100}{45}$ or $\frac{20}{9}$
$\frac{5}{6}$	$\frac{6}{5}$	-6.8	$-\frac{5}{34}$

2.


$-\frac{1}{3}$
$\frac{1}{7}$
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
$-\frac{1}{2}$
$\frac{1}{3}$
$\frac{3}{4}$
$\frac{4}{3}$

3.


-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
$-\frac{3}{4}$
$-\frac{1}{3}$
$\frac{1}{3}$
$\frac{3}{4}$
$-\frac{1}{2}$
$\frac{1}{2}$

[page 164]

In Problem 3 each double arrow connects points on opposite sides of 0, while in Problem 2 the points connected by a double arrow are both on the same side of 0.

4. If $a > 1$, then b is between 0 and 1.
   If a is between 0 and 1, then $b > 1$.
   1 is a multiplicative inverse of 1.

5. If $a < -1$, then b is between -1 and 0.
   If $a < 0$ and $a > -1$, then $b < -1$.
   -1 is a multiplicative inverse of -1.

6. If a is positive, the inverse b is positive.
   If a is negative, the inverse b is negative.

7-6. Multiplication Property of Equality

See comment on the addition property of equality in Chapter 6, page 155.

Answers to Problem Set 7-6; pages 165-166:

1. $(-5)(-3)$ and 15 are different names for the same number. When we multiply $(\frac{1}{3})$ by this number we obtain "$\big((-5)(-3)\big)(\frac{1}{3})$" and "$(15)(\frac{1}{3})$" as different names for a new number.

2. (a) true (b) true (c) false (d) true (e) true

3. (a) If $12x = 6$ is true for some x

$$\text{then } (12x)(\tfrac{1}{12}) = 6(\tfrac{1}{12}) \text{ is true for the same } x,$$
$$(12\cdot\tfrac{1}{12})x = 6(\tfrac{1}{12}) \text{ is true for the same } x,$$
$$x = \tfrac{1}{2} \text{ is true for the same } x.$$

If $x = \frac{1}{2}$.

the left member is $12(\frac{1}{2}) = 6$,

the right member is 6,

and hence the truth set is $\{\frac{1}{2}\}$.

(b) $\{\frac{6}{7}\}$

(c) $\{1\}$

(d) $\{3\}$

(e) $\{1\}$

(f) $\{\frac{2}{5}\}$

(g) $\{\frac{3}{2}\}$

(h) $\{1\}$

(i) $\{\frac{9}{4}\}$

4. The form used in Problem 3 may be used here also.

(a) $\{5\}$

(b) $\{35\}$

(c) $\{\frac{5}{7}\}$

(d) $\{-\frac{9}{2}\}$

(e) $\{15\}$

(f) $\{-\frac{9}{2}\}$

(g) $\{-4\}$

(h) $\{0\}$

5. (a) If V is 84 and h is 7, then $V = \frac{1}{3}Bh$ is $84 = \frac{1}{3}B(7)$.

If $84 = \frac{1}{3}B(7)$ is true for some B,

then $84 = B(7)(\frac{1}{3})$ is true for the same B,

$84 = B\cdot\frac{7}{3}$ is true for the same B,

$84\cdot\frac{3}{7} = B\cdot\frac{7}{3}\cdot\frac{3}{7}$ is true for the same B,

$36 = B\cdot(\frac{7}{3}\cdot\frac{3}{7})$ is true for the same B,

$36 = B\cdot 1$ is true for the same B,

$36 = B$ is true for the same B.

If B is 36,
the left member is 84,
the right member is $\frac{1}{3}(36)(7)$ or 84.

The truth set is $\{36\}$.

[page 166]

(b) The form of part (a) is to be used here. The truth set is {13}.

7-7. Solutions of Equations

Pages 167-169. This is our first use of "if and only if". We bring it in here because the situation is easily visualized in terms of truth sets. Although the idea is not difficult, "if and only if" often gives rise to confusion. The form always is "A if and only if B", where A and B are sentences. We are actually dealing with the compound sentence "A if B and A only if B". The sentence "A if B" is a compact way to write "If B then A", and "A only if B" is a way of writing "If A then B". These conditional sentences are sometimes written "B implies A" and "A implies B". Some writers abbreviate "if and only if" to "iff". The compound sentence then reduces to "A iff B". The confusion with "if and only if" comes from trying to remember which statement is the "if" statement and which is the "only if" statement. Everyone has this trouble but it is fortunately not an important matter. What is important is that the compound sentence "A if and only if B" means "If A then B and If B then A".

Equivalent sentences will be discussed in more detail in Chapter 13. You may wish to refer to the later discussion before taking it up at this point. The idea is introduced here for linear equations because the student must surely be aware of it by now and will become impatient with the checking routine. It is not our intention to do away with checking altogether for these equations, but rather to put it in its proper perspective - a check for errors in arithmetic.

It is important that the teacher note, and help the student note, that in the process of solving equations, not all steps involve directly the equivalence of two equations. Those steps in which the addition property and multiplication property of equality are used must raise the question of equivalence, but on the other hand there may be steps taken with the sole purpose of simplifying one member or both members of an equation.

Thus in going from

$$3x + 7 = x + 15$$

to $(3x + 7) + \big((-x) + (-7)\big) = (x + 15) + \big((-x) + (-7)\big)$,
equivalence is an issue because, for example, the phrase on the left names a number different from that named by the left member of the original equation, as the addition property for equality has been used. But in going from

$$(3x + 7) + \big((-x) + (-7)\big) = (x + 15) + \big((-x) + (-7)\big)$$

to

$$2x = 8$$

the question of equivalence does not enter the picture because all that is happening is that each member of the equation is being written in simpler form. Both types of steps are important, of course, and students should be able to give reasons for them.

A prolonged discussion in the text of the difference between these steps could have been a distraction to the main idea, and so the task of emphasizing the distinction is largely the teacher's. This is probably appropriate, because many natural opportunities to point this out will arise in class discussion throughout the course.

In connection with the work on equivalent equations, some teachers report that classes have found good practice and enjoyment as well in the process of building complicated equations from simple ones by use of equivalent equations. For example,

$$\begin{aligned} x &= 3 \\ x + 1 &= 4 \\ 2(x + 1) &= 8 \end{aligned} \qquad\qquad \begin{aligned} 2x + 2 &= 8 \\ 2x + 2 + 7 &= 8 + 7 \\ 2x + 9 &= 15 \end{aligned}$$

Answers to Problem Set 7-7; pages 170-172:

1. If x is a solution of $x + (-3) = 5$
then x is a solution of $x + (-3) + 3 = 5 + 3$ addition property of equality,
and $x = 8$ addition property of opposites and addition property of 0.

If x is a solution of $x = 8$

then x is a solution of $x + (-3) = 8 + (-3)$ addition property of equality.

and $x + (-3) = 5$ addition.

Thus the equations are equivalent equations, because the steps used in going from one equation to the other are reversible.

In a similar manner parts 1(b) through 1(i) may be done. 1(b), (c), (d), (f), (g), (h) are pairs of equivalent equations. 1(c) is not because multiplication by zero is not a reversible step, since zero has no multiplicative inverse. In 1(i) also the equations are not equivalent because taking the absolute value of each member of an equation is not a reversible step.

2. The example given in the text at the outset of this problem indicates the form to be used. Notice how "is equivalent to" is written between equations, one of which has been obtained from the other by use of the addition property or multiplication property of equality. Nothing is written between equations in which we have written a simpler numeral for one or both members of the equations. Reasons could be written for both sorts of steps, and we hope the student will be thinking of reasons for all steps. Here, though, the emphasis is on equivalence and the properties which yield equivalent sentences.

Have the student use this form only as long as you think it is necessary. After he stops writing "this sentence is equivalent to" you should check now and then with questions to make sure that he is thinking clearly about what he is doing. If he is not, you may want him to go back temporarily to writing the full form.

(a) $2a + 5 = 17$

This sentence is equivalent to

$$2a + 5 + (-5) = 17 + (-5).$$

$$2a = 12,$$

and this sentence is equivalent to

$$\frac{1}{2}(2a) = \frac{1}{2}(12).$$

$$a = 6$$

Hence the truth set is $\{6\}$.

(b) The truth set is the set of all real numbers.

(c) $\{6\}$

(d) $\{-3\}$

(e) $\emptyset$

(f) $\{0\}$

(g) $\{1\}$

(h) The set of all real numbers

(i) $\{-\frac{7}{3}\}$

(j) $\{-3\}$

(k) $\{0\}$

(ℓ) $\emptyset$

(m) $\{1\}$

3. (a) If x is the number of inches in the length of the third side,
then $2x + 3$ is the number of inches in the length of the second side,
and $x + 5$ is the number of inches in the length of the first side.

$$(x + 5) + (2x + 3) + x = 44$$

The truth set is $\{9\}$, and the sides of the triangle are 14, 21, and 9 inches in length.

(b) If m is the integer,
$m + 1$ is the successor.

$$m + (m + 1) = 2m + 1$$

The truth set is the set of all real numbers.

(c) If x is the first odd integer
then $x + 2$ is the second odd integer,
and $x + (x + 2) = 11$. The domain of x being the odd integers, the truth set is $\emptyset$, and thus there are

no two consecutive odd integers whose sum is 11.

(d) If the wire cost c cents per foot, then
Mr. Johnson paid $(30c + 55c)$ cents, and his neighbor paid $25c$ cents.

$$30c + 55c = 25c + 420$$

The truth set is $\{7\}$.
The wire cost 7 cents per foot.

(e) If the integer is n,
the successor is $n + 1$.

$$4n = 2(n + 1) + 10$$

The truth set is $\{6\}$, and the integer is 6.

(f) If each man was driving at the rate of r miles per hour,
then $5r$ is the number of miles the first man went.
$5r + 120$ was number of miles in the total length of the course.
$3r$ is the number of miles the second man went.
$3r + 250$ was the number of miles in the total length of the course.

$$5r + 120 = 3r + 250$$

The truth set is $\{65\}$, and each man was driving 65 miles an hour.

(g) If x is the number of weeks from now
then $2x$ is the number of inches of growth of plant A,
and $20 + 2x$ inches is the height of plant A, x weeks from now.
$3x$ is the number of inches of growth of plant B,
and $12 + 3x$ inches is the height of plant B, x weeks from now.

$$20 + 2x = 12 + 3x$$

$$\{8\}$$

In 8 weeks they will be equally tall.

(h) If x is the number,

$3(x + 17) = 192.$

[47]

The number is 47.

(i) If h is the smaller number,

then $5h$ is the larger.

$h + 5h = 4h + 15$

$[7\frac{1}{2}]$

The smaller number is $7\frac{1}{2}$.

(j) If x is the number of quarts of water,

then $x + 2$ is the number of quarts of mixture,

and $.20(x + 2)$ is the number of quarts of alcohol in the radiator.

$2 = .20(x + 2)$

[8]

There are 8 quarts of water in the radiator.

7-8. Reciprocals

We are now using the symbol "$\frac{1}{a}$" to represent the reciprocal of a. Before this, we have used the same symbol as a fraction, indicating a quotient. For our present purpose, it would have been nice to have a symbol which means only "reciprocal" and does not give away some of the later properties of the reciprocal. We considered using a' or a^* or a^{-1} or some such symbol temporarily, corresponding to our use of the "upper negative" early in Chapter 5, but decided that this would be more confusing than helpful. In addition, we feel that the student must become accustomed to symbols and words with more than one meaning. The particular meaning attached to such a symbol must be determined by the context.

Answers to Problem Set 7-8a; page 173:

1. (a) $\frac{1}{15}$

   (b) $\frac{1}{-8}$

   (c) $\frac{1}{\frac{1}{5}}$

   (d) $\frac{1}{-\frac{1}{6}}$

   (e) $\frac{1}{\frac{5}{3}}$

   (f) $\frac{1}{0.3}$

   (g) $\frac{1}{-\frac{3}{4}}$

2. (a) $\frac{1}{15}$

   (b) $-\frac{1}{8}$

   (c) 5

   (d) -6

   (e) $\frac{3}{5}$

   (f) $\frac{10}{3}$

   (g) $-\frac{4}{3}$

   In (a) the common name for the multiplicative inverse was the same as the reciprocal written in Problem 1.

3. For each non-zero real number a there is only one multiplicative inverse of a.

   Proof: Assume that both b and x are multiplicative inverses of a. Then

   $ab = 1$ and $ax = 1$. definition of multiplicative inverse.

   If $ax = 1$ for some x,

   $ax(b) = 1(b)$ for the same x. multiplication property of equality.

   $(ab)x = b$ commutative and associative properties of multiplication.

   $1(x) = b$ definition of reciprocal.

   $x = b$ multiplication property of 1.

Page 175. $\frac{1}{\frac{1}{a}}$ and a are both reciprocals of $\frac{1}{a}$ because when either is multiplied by $\frac{1}{a}$ the result is 1.

Answers to Problem Set 7-8b; pages 176-177:

1. The reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$, of 0.3 is $\frac{10}{3}$, of -0.3 is $-\frac{10}{3}$, of 0.33 is $\frac{100}{33}$, of -0.33 is $-\frac{100}{33}$, of 1 is 1, of -1 is -1, of $\sqrt{2}$ is $\frac{1}{\sqrt{2}}$, of $\frac{1}{x^2 + 4}$ is $x^2 + 4$, of $y^2 + 1$ is $\frac{1}{y^2 + 1}$.

2. $a + (-1)$ has no reciprocal if $a = 1$, since $a + (-1) = 0$ when $a = 1$, and 0 has no reciprocal.

   $a + 1$ has no reciprocal when $a = -1$.

   $a^2 + (-1)$ has no reciprocal when $a = 1$ or $a = -1$. Since $a^2 + (-1) = (a + 1)\big(a + (-1)\big)$, $a^2 - 1 = 0$ if and only if $a + 1 = 0$ or $a + (-1) = 0$, that is, if and only if $a = -1$ or $a = 1$.

   $a(a + 1)$ has no reciprocal for $a = 0$ or $a = -1$.

   $\frac{a}{a + 1}$ has no reciprocal for $a = 0$; it is not a number if $a = -1$.

   $a^2 + 1$ will have a reciprocal for any real a, for $a^2 + 1 \geq 1$ for all real numbers a.

   $\frac{1}{a^2 + 1}$ will have a reciprocal for any real a.

3. $$\Big(\big(a + (-3)\big)(a + 1)\Big)\left(\frac{1}{a + (-3)}\right) = \big(a + (-3)\big)\left(\frac{1}{a + (-3)}\right).$$

   Multiplication property of equality.

   $$(a + 1)\left(\big(a + (-3)\big)\frac{1}{a + (-3)}\right) = \big(a + (-3)\big)\frac{1}{a + (-3)}.$$

   Associative and commutative properties of multiplication.

   $(a + 1)\cdot 1 = 1.$ Definition of reciprocal.

   $a + 1 = 1.$ Multiplication property of 1.

   If $a = 3$, then $3 + 1 = 1$, and this is false.

We should not expect the sentence $a + 1 = 1$ to have the same truth set as the original sentence since our "multiplier" $\frac{1}{a + (-3)}$ is not a number when $a = 3$, and we used the multiplication property of equality in the very first step. In manipulating phrases, as in this example, we have to be constantly on guard that we do not become so engrossed in "pushing symbols" that we forget our algebraic structure. So long as we remember that $\frac{1}{a + (-3)}$ here is supposed to represent a number, we are safe in using algebraic properties. When we view $\frac{1}{a + (-3)}$ as a symbol only and apply our algebraic properties, any results we get can be only symbolic; to be interpreted as results about numbers, we have to check to see that we were actually using (symbolic) numbers at each step along the way.

4. Theorem 7-8b says "The reciprocal of a positive number is positive, and the reciprocal of a negative number is negative." Since the reciprocal of a real number is the multiplicative inverse of that number and the additive inverse of a number is its opposite, the corresponding statement for opposites would be "the opposite of a positive number is a negative number, and the opposite of a negative number is a positive number."

   Theorem 7-8c says "The reciprocal of the reciprocal of a non-zero real number a is a." The opposite of the opposite of a real number a is the number a

   Since 0 has no reciprocal, Theorem 7-4 does not apply to the case $a = 0$. The opposite of 0 is 0, and the opposite of the opposite of 0 is also 0. Thus, the restriction to non-zero real numbers is not needed in the statement concerning opposites.

5. If $a = 2$, $b = 3$ then $\frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6} = \frac{1}{2\cdot 3}$.

If $a = 4$, $b = -5$, then $\frac{1}{4} \cdot \frac{1}{-5} = \frac{1}{-20} = \frac{1}{4(-5)}$.

If $a = -4$, $b = -7$, then $\frac{1}{-4} \cdot \frac{1}{-7} = \frac{1}{28} = \frac{1}{(-4)(-7)}$.

Thus the sentence $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$ holds true in all three cases.

6. Is $\frac{1}{a} > \frac{1}{b}$ in all three cases?

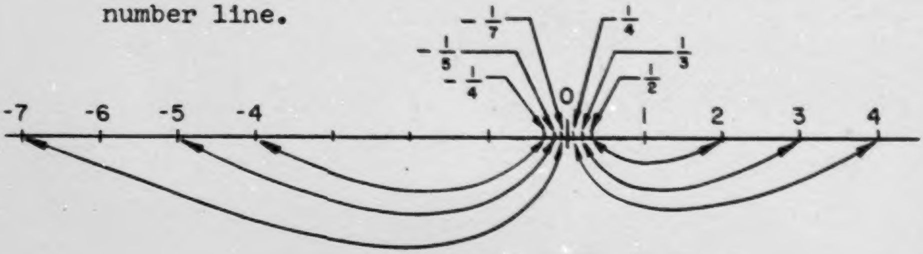
(a) $\frac{1}{2} > \frac{1}{3}$ is true.

$\frac{3}{6} > \frac{2}{6}$.

(b) $\frac{1}{4} > \frac{1}{-5}$ is true. One is positive, one negative.

(c) $\frac{1}{-4} > \frac{1}{-7}$ is false, since

$\frac{1}{-4}$, or $-\frac{7}{28}$, is to the left of $\frac{1}{-7}$, or $-\frac{4}{28}$, on the number line.



7. If $a > b$, a and b positive, then $\frac{1}{b} > \frac{1}{a}$. This is a true statement. An example:

If $a = 5$, $b = 2$, $(a > b)$

then $\frac{1}{2} > \frac{1}{5}$. $(\frac{1}{b} > \frac{1}{a})$

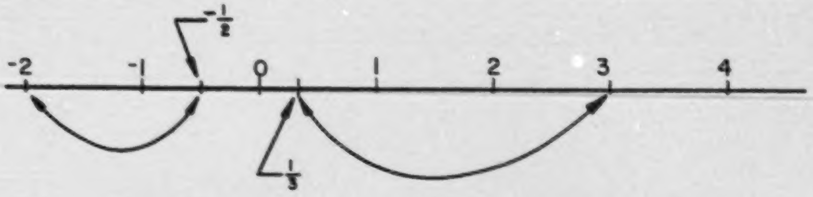
8. If $a > b$, a and b negative, then $\frac{1}{b} > \frac{1}{a}$. This is a true statement. An example:

If $a = -4$, $b = -8$, $(a > b)$

then $\frac{1}{-8} > \frac{1}{-4}$; hence $\frac{-1}{8}$ is to the right of $\frac{-1}{4}$ on the number line.

[page 176]

9. If a is positive and b is negative, $a > b$, then $\frac{1}{a} > \frac{1}{b}$ for the reciprocal of a positive number is a positive number, and the reciprocal of a negative number is a negative number. If $a = 3$, $b = -2$.



*10. We know that $a \cdot \frac{1}{a} = 1$ and we now assume a to be negative. Then by the comparison property exactly one of three possibilities must be true:

$$\frac{1}{a} < 0 \text{ , } \frac{1}{a} = 0 \text{ , } 0 < \frac{1}{a} \text{ .}$$

If $\frac{1}{a}$ is positive, then $a \cdot \frac{1}{a}$ is negative, by the definition of multiplication of real numbers. This contradicts the fact that $a \cdot \frac{1}{a}$ is positive. If $\frac{1}{a}$ is 0, when a is negative, then $a \cdot \frac{1}{a} = 0$ by the zero property of multiplication. This again contradicts the fact that $a \cdot \frac{1}{a}$ is positive. Hence the only possibility remaining is that $\frac{1}{a} < 0$.

Pages 177-178. In dealing with a "system" which has "structure", induction plays an important role because it helps us discover properties that might be true in general. Our confidence in the truth of the suggested property depends on the fact that we are dealing with a system. The proof of a property suggested by induction is made by showing that it follows from other properties which are known or assumed to be true. We discovered many of the basic properties of the real numbers by induction. At the outset we had to assume that the properties were true in general. Later we were able to prove that some of our discovered properties were consequences of properties which we had either clearly proved or assumed to be true. Thus all of our work rests on the *assumption* that certain properties of the real numbers are true in general. The realization of this fact may be disconcerting at first, but actually all of mathematics is developed in this way. When the assumed properties are separated out and stated explicitly, they are commonly called *Axioms* or *Postulates.*

From time to time we point out informally the difference between inductive reasoning and deductive reasoning. In some classes a more extended discussion of this may be worthwhile.

A discussion of deductive reasoning and references to pertinent literature can be found on pages 112 - 113 of the Appendices to the Report of the Commission on Mathematics. In the Review Problems of Chapter 14 is included the expression $a^2 - a + 41$, which is prime for all integral values of a from 1 to 40, but not for $a = 41$. This will provide a good opportunity to bring the inductive - deductive question before the students again.

Answers to Problem Set 7-8c; page 178:

1. (a) $\frac{1}{6ab}$ (c) $\frac{1}{21yz}$ (e) $\frac{1}{6m^3n^3}$

   (b) $\frac{1}{3ax^2}$ (d) $\frac{1}{3a^3b}$ (f) 1

2. 0

3. no, because $8 \cdot 17$ is 136, hence $8 \cdot 17$ is *not* 0

4. n is 0

5. p can be any real number

6. p is 0 or q is 0

7. q is 0

Pages 178-179. Impress once again on the students that the sentence "$a = 0$ or $b = 0$" allows the possibility of both a and b being 0.

When $a = 0$, the requirement that $a = 0$ or $b = 0$ is satisfied, because a compound sentence whose clauses are connected by "or", that is a disjunction, is true if at least one clause is true.

The reasons in the proof of Theorem 7-8e are:

$(\frac{1}{a})(ab) = \frac{1}{a}\cdot 0$	multiplication property of equality.
$(\frac{1}{a})(ab) = 0$	multiplication property of 0.
$\left(\frac{1}{a}\cdot a\right)b = 0$	associative property of multiplication.
$1\cdot b = 0$	definition of a reciprocal.
$b = 0$	multiplication property of 0.

Answers to Problem Set 7-8d; pages 179-180:

1. If $\left(x + (-5)\right)\cdot 7 = 0$, then $7 = 0$ or $x + (-5) = 0$ by Theorem 7-6. But 7 cannot equal 0, as they are names for different numbers.
   Thus $x + (-5) = 0$.

2. If $9 \times y \times 17 \times 3 = 0$, then by Theorem 7-8e, $9 = 0$ or $y = 0$ or $17 = 0$ or $3 = 0$. Since $9 \neq 0$, $17 \neq 0$, $3 \neq 0$, we must have $y = 0$.

*3. No. For example, -2 is between -3 and 1, but $-\frac{1}{2}$ is not between $-\frac{1}{3}$ and 1.

If, however, a, p, and q are all positive or all negative numbers and a is between p and q, then $\frac{1}{a}$ is between $\frac{1}{p}$ and $\frac{1}{q}$.

4. (a) If $\big(x + (-20)\big)\big(x + (-100)\big) = 0$ is true for some x, then $x + (-20) = 0$ or $x + (-100) = 0$ is true for the same x.

$$x = 20 \text{ or } \qquad x = 100$$

If $x = 20$, $\big(20 + (-20)\big)\big(20 + (-100)\big) = 0\big(20 + (-100)\big) = 0.$

If $x = 100$, $\big(100 + (-20)\big)\big(100 + (-100)\big) = \big(100 + (-20)\big)0 = 0.$

Hence, the truth set is $\{20, 100\}$.

(b) $\{-6, -9\}$

(c) $\{0, 4\}$

(d) $\{\frac{5}{3}, \frac{1}{2}\}$

(e) $\{1, 2, 3\}$

(f) $\{\frac{1}{2}, -\frac{3}{4}\}$

(g) $\{\frac{5}{3}, -\frac{1}{2}\}$

(h) $\{6\}$

(i) $\{2\}$

5. Theorem: If a, b, c are real numbers, and if $ac = bc$ and $c \neq 0$, then $a = b$

Since $ac = bc$ and $c \neq 0$,

then $ac(\frac{1}{c}) = bc(\frac{1}{c})$ multiplication property of equality.

$a\big(c(\frac{1}{c})\big) = b\big(c(\frac{1}{c})\big)$ associative property of multiplication.

$a \cdot 1 = b \cdot 1$ definition of reciprocal.

$a = b$ multiplication property of 1.

Page 181. If $\frac{1}{a} + \frac{1}{b}$ were equal to $\frac{1}{a + b}$ for a pair of numbers a and b, then we would have $a \neq 0$, $b \neq 0$, $a + b \neq 0$, and

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{a + b},$$

$b(a + b) + a(a + b) = ab.$ We have multiplied by $ab(a + b)$.

$$ab + b^2 + a^2 + ab = ab,$$

$$a^2 + ab + b^2 = 0.$$

We may now "complete the square" to get

$$(a^2 + ab + \tfrac{1}{4}b^2) + \tfrac{3}{4}b^2 = 0,$$

$$(a + \tfrac{1}{2}b)^2 + \tfrac{3}{4}b^2 = 0.$$

Since $x^2 = x \cdot x \geq 0$ for all real numbers x and since the sum of two non-negative numbers is non-negative, we must have $a + \frac{1}{2}b = 0$ and $b = 0$. Since $b \neq 0$, this is a contradiction and, so, there can be no real numbers a and b for which $\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}$.

Page 181. Proof of Problem 6.

$\frac{1}{a(-1)} = \frac{1}{a} \cdot \frac{1}{-1}$	$\frac{1}{xy} = \frac{1}{x} \cdot \frac{1}{y}$
$= \frac{1}{a}(-1)$	Definition of reciprocal: $(-1)(-1) = 1$.
$= (-1) \cdot \frac{1}{a}$	Commutative property of multiplication.
$= -(\frac{1}{a})$	$(-1)x = -x$

"The opposite of the sum of two numbers is the sum of their opposites" and "the reciprocal of the product of two numbers is the product of their reciprocals" are closely parallel statements.

Answers to Review Problems; pages 182-183:

1. Summary:

We have defined the *product* of two real numbers a and b as follows:

> If a and b are both negative or both non-negative, then $ab = |a|\,|b|$.
>
> If one of the numbers a and b is positive or zero, and the other is negative, then $ab = -(|a|\,|b|)$.

From this definition and the properties of operations upon numbers of arithmetic, the following properties of operations can be proved:

1. *Multiplication property of 0:* For any real number a,

$$(a)(0) = 0 .$$

2. *Multiplication property of 1:* For any real number a,

$$(a)(1) = a.$$

3. *Commutative property of multiplication:* For any real numbers a and b,

$$ab = ba.$$

4. *Associative property of multiplication:* For any real numbers a, b, c,

$$(ab)c = a(bc).$$

5. *Distributive property:* For any real numbers, a, b, and c,

$$a(b + c) = ab + ac.$$

We have defined a *multiplicative inverse* as follows:

If c and d are real numbers such that

$$cd = 1,$$

then d is a multiplicative inverse of c.

We have agreed to use the name *reciprocal* for the multiplicative inverse, and to represent the reciprocal of any number a by the symbol "$\frac{1}{a}$".

We have proved the following theorems:

1. The number 0 has no reciprocal.
2. For each non-zero real number a there is only one multiplicative inverse of a.
3. The reciprocal of a positive number is positive, and the reciprocal of a negative number is negative.
4. The reciprocal of the reciprocal of a non-zero real number a is a.
5. For any non-zero real numbers a and b,

   $$\frac{1}{a}\cdot\frac{1}{b} = \frac{1}{ab}.$$

6. For real numbers a and b, $ab = 0$ if and only if $a = 0$ or $b = 0$.

We have stated the multiplication property of equality: For any real numbers a, b, and c, if $a = b$, then $ac = bc$.

2. (a) $3a^2 + (-6a)$ (d) $m^2 + (-10m) + 25$
   (b) $x^2 + 7x + 6$ (e) $2x^2 + (-5x) + (-12)$
   (c) $a^2 + (-b^2)$

3. (a) $2a(x + y)$ (d) $5\left(2x^2 + (-3x) + (-1)\right)$
   (b) $c\left(a + (-b) + 1\right)$ (e) $3x\left(3x^2 + 2x + (-1)\right)$
   (c) $(a + b)(cx + y)$

4. (a) $\{2\}$ (d) $\{0\}$
   (b) $\{7\}$ (e) $\{-1\}$
   (c) set of all real numbers

5. (a) $4a + 3b$ (c) $(-6a) + 4.6$
   (b) $4x + (-2b)$ (d) $|x| + |-x|$

*6. (a)

x	-2	-1	0	2
-2	4	2	0	-4
-1	2	1	0	-2
0	0	0	0	0
2	-4	-2	0	4

x	-4	-2	0	1	2	4
-2	8	4	0	-2	-4	-8
-1	4	2	0	-1	-2	-4
0	0	0	0	0	0	0
2	-8	-4	0	2	4	8

$P = \{-8, -4, -2, -1, 0, 2, 4, 8\}$

(b)

+	-2	-1	0	2
-2	-4	-3	-2	0
-1	-3	-2	-1	1
0	-2	-1	0	2
2	0	1	2	4

+	-4	-3	-2	-1	0	1	2	4
-2	-6	-5	-4	-3	-2	-1	0	2
-1	-5	-4	-3	-2	-1	0	1	3
0	-4	-3	-2	-1	0	1	2	4
2	-2	-1	0	1	2	3	4	6

$R = \{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 6\}$

7. (a) If I pay x dollars,

Jim pays $x + 2$ dollars.

$$x + (x + 2) = 11$$

$$\{4\tfrac{1}{2}\}$$

Jim pays $6.50.

(b) If the first odd integer is x,

the next odd integer is $x + 2$.

$$x + (x + 2) = 41$$

$\emptyset$, since the domain of x is the odd integers. There are not two consecutive odd integers such that their sum is 41.

(c) If the number of yards in the width of the rectangle is w, then the number of yards in the length of the rectangle is $w + 27$.

$$2w + 2(w + 27) = 398$$

$$\{86\}$$

The rectangle is 113 yards long and 86 yards wide.

(d) If Jim's grade is x,

Mary's grade is $x + 14$.

$$x + (x + 14) = 170$$

$$\{78\}$$

Jim's grade was 78, Mary's grade was 92.

(e) If the father's wage was x dollars per day,

the son's wage was $(\frac{2}{5})x$ dollars per day.

The son earned $2(\frac{2}{5})x$ dollars.

The father earned $4x$ dollars.

$$4x + 2(\tfrac{2}{5})x = 96$$

$$\{20\}$$

The father earned 20 dollars a day, and the son earned 8 dollars a day.

(f) If x is the number of pigs,
$4x$ is the number of pigs' feet,
$x + 16$ is the number of chickens,
$2(x + 16)$ is the number of chickens' feet.
$4x + 2(x + 16) = 74$
[7]
There were 7 pigs and 23 chickens.

(g) If x is the number of hits,
$x + 10$ is the number of misses.
$10x + (-5)(x + 10) = -25$
[5]
He made 5 hits.

Chapter 7

Suggested Test Items

1. Find the value of each of the following when x is -3, y is 2, a is -4, and b is $\frac{1}{2}$.
   (a) $2ax + 3by$
   (b) $2ab\big((-x) + y\big)$
   (c) $\big(a + x + (-y)\big)^2$
   (d) $2x + (-a) + (-y^2)$
2. Multiply the following:
   (a) $(-7)(3x + 4y)$
   (b) $3y\big(y^2 + (-2xy) + x^2\big)$
   (c) $(x + 6)(x + 7)$
   (d) $\big(5x + (-2)\big)(3x + 7)$
   (e) $\big(x + (-4)\big)\big(x^2 + 5x + (-3)\big)$
   (f) $(\frac{1}{2}x^2)(-6xy)$
   (g) $(\frac{1}{2a})(\frac{1}{5a^2b})$
3. Write each of the following as an indicated product.
   (a) $7a + 7b$
   (b) $3m + 15n$
   (c) $4p + (-7px)$
   (d) $xy + (-xy) + (-x)$
   (e) $(-4)a^2 + (-4)x^2$

4. Collect terms in the following:

(a) $z + 3z$ (c) $4x + (-6y) + 6x + 12y$

(b) $(-15a) + a$ (d) $x + 3y + 7x + (-2y) + 4y$

5. For what values of b does each of the following have no reciprocal:

(a) $\frac{a}{b}$ (d) $\frac{2a + b}{b^2 + 1}$

(b) $\frac{2a}{b - 1}$ (e) $\frac{2a + b}{b(b + 1)}$

(c) $\frac{2a + b}{(b - 1)(b + 1)}$

6. The following sentences are true for every a, every b, and every c :

A. $ab = ba$

B. $(ab)c = a(bc)$

C. $a(1) = a$

D. $a(0) = 0$

E. $(-a)(-b) = ab$

F. If $a = b$, then $ac = bc$.

G. $a(b + c) = ab + ac$

Which of the sentences expresses:

(a) The associative property of multiplication

(b) The distributive property

(c) The multiplication property of order

(d) The multiplication property of one

7. Find a real number x to make each open sentence true:

(a) $\frac{1}{3}x + (-8) = 4$ (d) $|x + (-1)| = 4(-3) + (-2)(-8)$

(b) $|-5| + 7 + (-5) + 2x = 0$ (e) $2x + 3x = 8 - 3x$

(c) $-\big((-5)x + 7\big) = 5x + (-7)$

8. (a) If a and b are real numbers, state the property used in each step of the following:

$$\begin{aligned}(a + b)\Big(a + (-b)\Big) &= (a + b)a + (a + b)(-b)\\ &= a^2 + ab + a(-b) + (-b^2)\\ &= a^2 + 1ab + a(-1)b + (-b^2)\\ &= a^2 + \Big(1 + (-1)\Big)ab + (-b^2)\\ &= a^2 + (0)ab + (-b^2)\\ &= a^2 + (-b^2)\end{aligned}$$

(b) State in words the fact about the real number a and b shown in the sentence

$$(a + b)\Big(a + (-b)\Big) = a^2 + (-b^2).$$

9. Find truth sets for the following open sentences and draw their graphs:

(a) $7r + 4 + 3r = (-4r) + 18$

(b) $4(y + 2) + (-6)(y + 3) + (-y) = -4$

(c) $4|x| = 18 + (-2|x|)$

(d) $3\Big(x + (-4)\Big)\Big(x + (-1)\Big) = 0$

(e) $x(x + 2) = 0$

10. Write an open sentence for each of the following problems. State the truth sets and answer the questions.

(a) Two automobiles 360 miles apart start toward each other at the same time and meet in 6 hours. If the rate of the first car is twice that of the second car, what is the rate of each?

(b) Four times a certain integer is two more than three times its successor. What is the integer?

(c) The perimeter of a triangle is 40 inches. The second side is 3 inches more than the first side, and the third side is one inch more than twice the first side. Find the length of each side.

Chapter 8

PROPERTIES OF ORDER

A general reference for teachers for this chapter is *Studies in Mathematics*, Volume III, Chapter 3, Section 3.

8-1. *The Order Relation for Real Numbers*

Remember that the main object of our attention in this chapter is the *order relation*. (See the introductory remarks in the Commentary on Chapter 6.) In talking about a given pair of numbers we may, and frequently do, shift from "less than" to "greater than" and back again without trouble. However, this tends to obscure the idea of order relation and is *not* permissible when we are studying the order relation "<" itself. We are making a big issue of this matter because it is mathematically important for the student to begin thinking of *order relation* and not just order. On the other hand, do not belabor the point with the students. It is not essential that they be able to explain it, etc. If *you* think about the order relation and are careful to discuss it correctly, then the student will automatically learn to think about an order relation as a mathematical object rather than as a convenient way of discussing a pair of real numbers.

Answers to Problem Set 8-1; pages 186-187:

1. (a) $-\frac{3}{2} < -\frac{4}{3}$

   (b) $-|-7| = -|7|$

   (c) Cannot tell; if c is a real number, then exactly one of the following is true: $c < 1$, $c = 1$, $1 < c$.

2. If $c > 4$ and $4 > 1$, then $c > 1$; transitive property.

3. (a) true

   (b) true

   (c) true

   (d) true for any real number a

4. (a) true (c) false
   (b) false (d) truth set consists of all positive numbers.

5. (a) false (c) true
   (b) false (d) true

6. (a) $3 < 3 + x$
   $0 < x$
   Three possible descriptions of the truth set are:
   The set of numbers x such that $0 < x$
   All numbers greater than 0
   The set of all positive numbers

   (b) $3 + x < 3$
   $x < 0$
   The set of all negative numbers

7. (a) $\{1\}$ (c) $\{1\}$
   (b) $\{5\}$ (d) $\{\pi + (-\sqrt{2})\}$

8. (a) the set of all numbers less than 3
   (b) the set of all numbers greater than -3 and less than 3
   (c) all real numbers
   (d) the set of all numbers greater than -3

8-2. Addition Property of Order

Addition is naturally tied in with order by the fact that $x + y$ is obtained on the number line by going to the right from x if y is positive and going to the left if y is negative. However, this is not the property which we want to emphasize. The property which is regarded as basic is the addition property of order. It is easy to prove that the first property mentioned above follows from the basic one. For example, if $a = 0$, then the condition $a < b$ says that b is positive. The addition property gives $c < b + c$, which says that, if b is positive then $b + c$ is to the right of c, etc.

[pages 186-188]

Theorem 8-1. Here is another result which is "discovered" by induction. The proof is another example of a proof by contradiction.

Page 188. $(-3) + (-3) < (-\frac{1}{2}) + (-3)$ is a true sentence. If c has successively the values $\frac{1}{2}$, 0, -7, the sentence is true in each case. A statement of the addition property of order in words is: "If one number is less than another, and to each of these is added the same number, the order remains unchanged." The corresponding property of equality is the addition property of equality.

Answers to Problem Set 8-2a; pages 188-190:

1. (a) true (c) true

   (b) true (d) true

2. If a, b, c, are real numbers and if $a \leq b$, then $a + c \leq b + c$.

   If a, b, c are real numbers and if $a > b$, then $a + c > b + c$.

   If a, b, c are real numbers and if $a \geq b$, then $a + c \geq b + c$.

3. If $a < b$, then $a + c < b + c$. addition property of order
   If $c < d$, then $b + c < b + d$. addition property of order
   Hence, $a + c < b + d$. transitive property

4. In the following example, call the students' attention to the fact that we could not test the truth set by substituting numbers into the original sentence, but had, instead, to verify the truth set by reversing the steps which we took in obtaining it. The students have seen this reversal before, in Chapter 7, Section 7-7, in connection with equations. At that point we learned to recognize that certain equations are equivalent *because* we *know* that the steps we took to obtain one from the other *are* reversible. Soon (page 198) we shall apply to inequalities this same procedure of recognizing equivalent sentences by the fact that the intervening steps are reversible.

(a) If $3 + x < (-4)$ is true for some x,
then $x < (-4) + (-3)$ is true for the same x,
and $x < -7$ is true for the same x.
Thus, if x is a number which makes the original sentence true, then $x < -7$.

If $x < -7$ is true for some x,
then $3 + x < (-7) + 3$ is true for the same x,
and $3 + x < (-4)$ is true for the same x.
Hence the truth set is the set of all real numbers less than -7.

The students should follow the above form in finding truth sets of this problem. The truth sets are as follows:

(b) The set of all numbers greater than -1

(c) The set of all x such that $x < -5$

(d) The set of all numbers greater than $\frac{4}{3}$

(e) The set of all x such that $x \geq \frac{7}{3}$

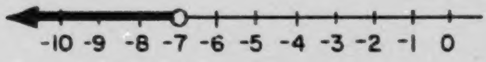
(f) The set of all numbers greater than 4

(g) The set of all x such that $-7 < x$

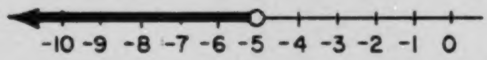
(h) The set of all numbers greater than -4

(i) The set of all x such that $x \leq -1$

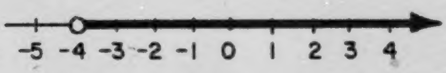
5. (a)



(c)



(h)



6. Theorem: If $a < b$, then $-b < -a$.

Proof: If $a < b$,	assumption
then $a + \big((-a) + (-b)\big) < b + \big((-a) + (-b)\big)$.	addition property of order
$\big(a + (-a)\big) + (-b) < \big(b + (-b)\big) + (-a)$.	commutative and associative properties of addition
$0 + (-b) < 0 + (-a)$	addition property of opposites
$-b < -a$	addition property of zero

*7. Theorem: If $0 < y$, then $x < x + y$.

Proof: For every a, b, c, if $a < b$,	
then $a + c < b + c$.	addition property of order
let $a = 0$, $b = y$, $c = x$:	
If $0 < y$, then $0 + x < y + x$.	
If $0 < y$, then $x < y + x$.	addition property of zero
If $0 < y$, then $x < x + y$.	commutative property of addition

Page 191. We may write the addition property of order thus:

"If $a < b$, then $c + a < c + b$"

because of the commutative property of addition.

Page 191. The sentence $4 = (-2) + 6$ can be changed to this sentence involving order: $(-2) < 4$.

The number y such that $7 = 5 + y$ is 2, found by the addition property of equality. In the same manner, the truth set of $-3 = -6 + y$ is found to be 3, again a positive number.

Page 192. The reasons for Theorem 8-2b are as follows.

If $y = z + (-x)$,	
then $x + y = x + \big(z + (-x)\big)$	addition property of equality
$= \big(x + (-x)\big) + z$.	associative and commutative properties of addition

In the next paragraph, y is negative, y is 0, or y is positive. Exactly one of these is true, by the comparison property.

Answers to Problem Set 8-2b; pages 193-195:

1. (a) $-24 < -15$, $-24 + 9 = -15$, $b = 9$

   (b) $-\frac{5}{4} < \frac{63}{4}$, $-\frac{5}{4} + \frac{68}{4} = \frac{63}{4}$, $b = \frac{68}{4} = 17$

   (c) $\frac{7}{10} < \frac{6}{5}$, $\frac{7}{10} + \frac{5}{10} = \frac{6}{5}$, $b = \frac{5}{10} = \frac{1}{2}$

   (d) $-\frac{1}{2} < \frac{1}{3}$, $-\frac{3}{6} + \frac{5}{6} = \frac{2}{6}$, $b = \frac{5}{6}$

   (e) $-345 < -254$, $-345 + 91 = -254$, $b = 91$

   (f) $-\frac{33}{13} < -\frac{98}{39}$, $-\frac{99}{39} + \frac{1}{39} = -\frac{98}{39}$, $b = \frac{1}{39}$

   (g) $-0.21 < 1.47$, $-0.21 + 1.68 = 1.47$, $b = 1.68$

   (h) $\left(\frac{3}{2}\right)\left(-\frac{5}{4}\right) < \left(-\frac{2}{3}\right)\left(\frac{4}{5}\right)$, $\left(-\frac{225}{120}\right) + \left(\frac{161}{120}\right) = \left(-\frac{64}{120}\right)$, $b = \frac{161}{120}$

2. We must show there _is_ a number b such that $c = a + b$, and then show that b is negative.

   If b is $c + (-a)$,

   then $b = c + (-a)$.

   $a + b = a + \big(c + (-a)\big)$ addition property of equality

   $= \big(a + (-a)\big) + c$ commutative and associative properties of addition

   $= 0 + c$ addition property of opposites

   $a + b = c$ addition property of zero

   Hence there is a number b such that $c = a + b$.

   Exactly one of these is true:

   $0 < b$, $b = 0$, or $b < 0$.

   If $0 < b$,

   then $a < a + b$. addition property of order

   $a < c$, since $c = a + b$.

   But $a < c$ and $c < a$ is a contradiction.

   If $b = 0$,

   then $a + b = a$. addition property of zero

   $c = a$, since $c = a + b$.

   But $c = a$ and $c < a$ is a contradiction.

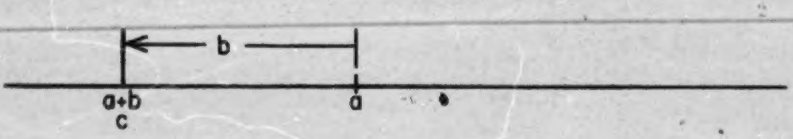
   Therefore, $b < 0$.

[page 193]

3. (a) true (d) true
(b) false (e) true
(c) true (f) true

4. (a) $5 < 13$, $8 < 13$, (also $5 < 8$)
(b) $(-3) < (-1)$, $(-1) < 2$, $(-3) < 2$
(c) $5 + 2 = 7$, $5 = 7 + (-2)$

5.



Locate a. Since $b < 0$, we move $|b|$ units to the left to locate $a + b$. c is another name for $a + b$. c is to the left of a, so $c < a$.

6. (a) false (b) true (c) true

7. (a) $|3 + 4| = |7| = 7$
$|3| + |4| = 3 + 4 = 7$ Thus, $|3 + 4| \leq |3| + |4|$ is true.
(b) $|(-3) + 4| = |1| = 1$
$|-3| + |4| = 3 + 4 = 7$ Thus, $|(-3) + 4| \leq |-3| + |4|$ is true.
(c) $|(-3) + (-4)| = |-7| = 7$
$|-3| + |-4| = 3 + 4 = 7$ Thus, $|(-3) + (-4)| \leq |-3| + |-4|$ is true.
(d) For any real numbers a and b,
$|a + b| \leq |a| + |b|$.

8. For any real numbers a and b,
$|a \cdot b| = |a| \cdot |b|$.

9. (a) If x is the number,
then $x + 5 < 2x$.
If $x + 5 < 2x$ is true for some x,
then $5 < x$ is true for the same x.

Thus, if x is a number which makes the original sentence true, then $5 < x$.

If $5 < x$,

then $x + 5 < 2x$ is true for the same x.

Hence the truth set is the set of all numbers greater than 5.

(b) If x is the number of dollars Moe would pay,

then $x + 130$ is the number of dollars Joe would pay,

and $x + x + 130 \leq 380$.

The truth set is the set of all numbers less than or equal to 125.

Thus Moe would pay no more than $125.

(c) If n is the number,

then $6n + 3 > 7 + 5n$.

The truth set is the set of all numbers greater than 4.

The number is any number greater than 4.

(d) If y is the number of students in the class,

then $2y \geq y + 26$.

The truth set is the set of all numbers greater than or equal to 26.

The number of students in class is at least 26.

*(e) If s is the score the student must make on the third test, then

$$\frac{82 + 91 + s}{3} \geq 90,$$

and $$82 + 91 + s \geq 3\,(90).$$

The truth set is the set of all numbers greater than or equal to 97.

The student must score at least 97 on the third test.

There may arise in this problem a question of the domain of s. Some students will claim a domain of $0 \leq S \leq 100$ is implied, some may assert that s may be any number, and some may ask "What about 30π?", etc. It is a good opportunity to note the importance of thinking about the domain of the variable.

[page 195]

*(f) If n is the number of years in Norman's age,

then $n + 5$ is the number of years in Bill's age,

and $n + 5 + n < 23$.

The truth set is the set of all numbers less than 9.

Norman is younger than 9 years old.

8-3. Multiplication Property of Order

The fundamental property is given in Theorem 8-3a. We "discovered" it by induction, but it is provable. This shows that the multiplication property of order is not independent of the other properties but can be deduced from them. It comes from the addition property of order by way of the distributive property which links addition and multiplication.

Page 196. Since $\frac{1}{4} < \frac{2}{7}$,

$\frac{1}{4}(5) < \frac{2}{7}(5)$ if $a < b$ and $c > 0$, $ac < bc$

and $\frac{5}{4} < \frac{10}{7}$.

Similarly, since $-\frac{5}{6} < -\frac{14}{17}$,

$\left(-\frac{14}{17}\right)\left(-\frac{1}{3}\right) < \left(-\frac{5}{6}\right)\left(-\frac{1}{3}\right)$ if $a < b$ and $c < 0$, $bc < ac$

and $\frac{14}{51} < \frac{5}{18}$.

Again, since $\frac{5}{3} < \frac{7}{4}$,

$\frac{7}{4}\left(-\frac{1}{4}\right) < \frac{5}{3}\left(-\frac{1}{4}\right)$ if $a < b$ and $c < 0$, $bc < ac$

and $-\frac{7}{16} < -\frac{5}{12}$

Page 197. The reasons for the steps of the proof of the first case of Theorem 8-3a are as follows:

1. If x and z are two real numbers such that $x < z$, then there is a positive number y such that $z = x + y$.
2. Multiplication property of equality.

3. Distributive property.

4. The product of two positive numbers is positive.

5. If $z = x + y$ and y is a positive number, then $x < z$.

The students may notice that this is an example of a proof done by translation back and forth between statements about order and statements about equality.

Pages 197-198. The multiplication property of order, seen from the standpoint of the numbers rather than the order relation, could be stated: If one number is less than another, and both are multiplied by a positive number, the order remains unchanged; if one number is less than another number, and both are multiplied by a negative number, the order is reversed.

The reasons for the steps of the proof of Theorem 8-3b are as follows:

If $x > 0$,
$(x)(x) > 0\,(x)$. multiplication property of order (If $a < b$, then $ac < bc$ if c is positive.)
If $x < 0$,
$(x)(x) > (0)(x)$. multiplication property of order (If $a < b$, then $bc < ac$ if c is negative)
If $x = 0$, $x^2 = 0$; and if $x \neq 0$, $x^2 > 0$.
Hence, for all real numbers x, $x^2 \geq 0$.

Page 198. To the members of $-8x < -8$ we must add $5x + 2$ to obtain the original sentence, $(-3x) + 2 < 5x + (-6)$.

Multiplying the members of $1 < x$ by -8, we obtain the sentence $-8x < -8$.

Answers to Problem Set 8-3; pages 199-200:

1. (a) the set of all numbers greater than 4
   (b) the set of all numbers less than 1
   (c) the set of all x such that $x < -4$

(d) the set of all numbers greater than $\frac{4}{3}$

(e) the set of all negative numbers

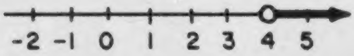
(f) the set of all x such that $x > \frac{3}{2}$

(g) the set of all numbers less than $\frac{17}{6}$

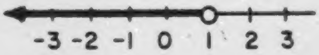
(h) the set of all numbers greater than $-\frac{4}{5}$

(i) the set of all numbers greater than 2

2. (a)



(b)



3. (a) If Sally has b books, then Sue has $b + 16$ books and

$$b + 16 + b > 28.$$

The truth set consists of all numbers greater than 6 so that Sally has at least 6 books.

(b) If x bulbs were planted, then

$$15 < \frac{5}{8}x \quad \text{and} \quad 15 > \frac{3}{8}x.$$

The truth set consists of all integers between 24 and 40.

4. If $a < b$, then $a = b + e$, where $e < 0$. (See Problem Set 8-2b, Problem 2.)

Then, if $c < 0$, $ac = (b + e)c$. multiplication property of order

$ac = bc + ec$ distributive property

If $c < 0$, then ec is positive, and $bc < ac$. If $z = x + y$ and y is positive, then $x < z$.

*5. If $c < 0$, then $0 < (-c)$, and $(-c)$ is positive.
If $a < b$, then $a(-c) < b(-c)$,
$-(ac) < -(bc)$,
and $bc < ac$. If $a < b$, then $-b < -a$.

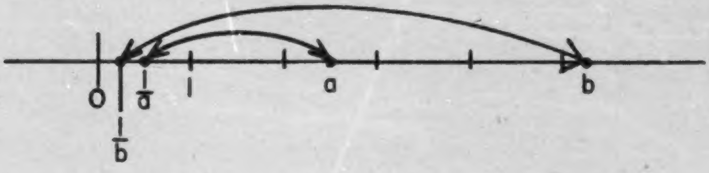
6. If $a < b$, $a > 0$ and $b > 0$,
then

$$a\left(\frac{1}{a}\cdot\frac{1}{b}\right) < b\left(\frac{1}{a}\cdot\frac{1}{b}\right).$$

$$\left(a\cdot\frac{1}{a}\right)\left(\frac{1}{b}\right) < \left(b\cdot\frac{1}{b}\right)\left(\frac{1}{a}\right)$$

$$(1)\left(\frac{1}{b}\right) < (1)\left(\frac{1}{a}\right)$$

$$\frac{1}{b} < \frac{1}{a}$$



7. If $a < b$, where $a < 0$ and $b < 0$,
then

$$\left(\frac{1}{a}\cdot\frac{1}{b}\right) \text{ is positive.}$$

$$a\left(\frac{1}{a}\cdot\frac{1}{b}\right) < b\left(\frac{1}{a}\cdot\frac{1}{b}\right)$$

$$\left(a\cdot\frac{1}{a}\right)\left(\frac{1}{b}\right) < \left(b\cdot\frac{1}{b}\right)\left(\frac{1}{a}\right)$$

$$\frac{1}{b} < \frac{1}{a}$$

The relationship holds when both a and b are negative.

8. If $a < b$, $a < 0$, $b > 0$, then $\frac{1}{a} < 0$ and $\frac{1}{b} > 0$; hence $\frac{1}{a} < \frac{1}{b}$, by the transitive property.

The relation $\frac{1}{b} < \frac{1}{a}$ does not hold if $a < 0$ and $b > 0$.

This can also be disproved by one "counter example":
If $a = -2$ and $b = 2$, then $\frac{1}{a} = -\frac{1}{2}$ and $\frac{1}{b} = \frac{1}{2}$.
Since $-\frac{1}{2} < \frac{1}{2}$, the relation is false.

9. Addition property of order: if a, b and c are real numbers, and if $a > b$, then $a + c > b + c$.

Multiplication property of order: if a, b and c are real numbers, and if $a > b$, then

$$ac > bc, \text{ if } c \text{ is positive;}$$
$$bc > ac, \text{ if } c \text{ is negative.}$$

10. If $a > 0$ and $b > 0$ and $a < b$, then by the multiplication property of order, we have

$$a \cdot a < a \cdot b \quad \text{and} \quad a \cdot b < b \cdot b.$$

If $a^2 < ab$ and $ab < b^2$,

then $a^2 < b^2$, by the transitive property of order.

8-4. The Fundamental Properties of Real Numbers

This section contains some of the most important mathematical ideas in the whole course. Up until now, we have taken the real numbers for granted as familiar objects and have occupied ourselves with "discovering" (and sometimes proving) properties about them. In this section we are trying to shift the students thinking about real numbers from a primarily inductive to a deductive point of view. The deductive point of view has, of course, been uppermost in our own minds from the beginning and has had a great deal of influence in our study of the real numbers. Thus we have consistently worked toward the list of fifteen fundamental properties listed here. Now that the list is complete (for our purposes), we can let the student in on the "secret". The idea to which the student should be exposed here is that we could have started with these properties in the first place by considering them, not as a description of, but rather as a definition of the real number system. Thus, "real number", "addition", "multiplication" and "order relation" become undefined terms, the fundamental properties become axioms (for an ordered field), and all other properties, as consequences of the axioms, become theorems. This is the axiomatic approach which becomes commonplace in more advanced mathematics.

Although we have given considerable preparation for the shift in point of view indicated here, most students obviously will not be able to appreciate fully its significance at this time. We have, however, no intention of proving everything from here on, but will continue as we have in the past, "discovering" properties and giving an occasional simple proof. The principal change is in our attitude toward the properties discovered, namely, that they could be proved if we had the time and experience to do so.

The best students will get the idea easily and we hope that most students, by the end of the course, will be able to think of the real number system deductively. The traditional geometry course, as well as the SMSG geometry course, requires a deductive approach to geometry. The SMSG geometry course also assumes the real numbers to be given axiomatically. Therefore, the deductive point of view is important not only for advanced mathematics but also for later high school courses.

Page 202. The missing fundamental property, which would enable us to obtain everything about the real numbers, is called the completeness axiom. It can be stated in a number of forms, one of the most convenient being in terms of "least upper bounds". Before stating it, we first define an "upper bound" of a set as follows:

> Let S be a set of real numbers and b a real number such that $s \leq b$ for every s in S. Then b is called an upper bound for S. If there does not exist an upper bound for S which is less than b, then b is called a least upper bound for S.

We can now state the

> Completeness Axiom. If S is any set of real numbers for which there is an upper bound, then there exists a least upper bound for S.

The completeness axiom is needed, for example, to prove the existence of $\sqrt{2}$. In other words, one can not prove, using only the fifteen properties stated above, that there is a real number a for which $a^2 = 2$. As a matter of fact, the completeness axiom

[pages 200-205]

is required to prove the existence of any irrational number. See Studies in Mathematics, III, Chapter 5, for a more detailed discussion of these ideas.

Answers to Review Problems; pages 206-208:

1. (a) $-100 < -99$
   (b) $0.2 > 0.1$
   (c) $|-3| < |-7|$
   (d) $\frac{6}{7} > \frac{5}{8}$
   (e) $3 \cdot 4 + (-4) > 3\big(4 + (-4)\big)$
   (f) $x^2 + 1 > 0$

2. (a) true
   (b) false
   (c) true
   (d) false
   (e) false
   (f) true

3. (a) not equivalent
   (b) equivalent
   (c) equivalent
   (d) equivalent
   (e) not equivalent
   (f) not equivalent

4. (a) positive
   (b) positive
   (c) negative
   (d) negative
   (e) positive
   (f) positive

5. (a) the set of all numbers less than (-5)
   (b) the set of all numbers greater than (-1)
   (c) the set of all numbers greater than (-6)
   (d) the set of all numbers less than (-3)
   (e) the set of all numbers less than or equal to 91
   (f) the empty set

6. (a) the set of all numbers greater than 2
   (b) $\{2\}$
   (c) the set of all negative real numbers
   (d) the set of all real numbers except zero
   (e) the set of all non-negative numbers less than 90
   (f) $\emptyset$

7. (a) $\{2\}$ (d) $\emptyset$
(b) $\{-1\}$ (e) the set of integers less than -2
(c) $\{-2\}$ (f) the set of integers greater than -1

8. (a) $\{\frac{5}{3}\}$ (d) $\{-1\}$
(b) $\{2\}$ (e) $\{0\}$
(c) $\{\frac{2}{3}\}$ (f) $\{-\frac{7}{3}\}$

9. (a) $\{-12\}$ (d) $\{0\}$
(b) $\{-3\}$ (e) the set of real numbers
(c) $\{0\}$ (f) $\{-\frac{5}{6}\}$

10. If A is the number of square units in the area,

$$24 \leq A < 28.$$

11. If A is the number of square units in the area,

$$24 \leq A < 35.$$

*12. If A is the number of square units in the area,

$$25.5225 \leq A < 26.5625.$$

13. (a) If p is the number of plants at the beginning of the second year,

$$p > \frac{3}{4}(240) \quad \text{and} \quad p < \frac{5}{6}(240);$$

that is, $180 < p < 200$.

If n is the number of seeds at the end of the second year, $n > (180)(240)$ and $n < (200)(240)$;
that is, $43{,}200 < n < 48{,}000$.

(b) If s is the number of seeds at the end of the second year, $s > (180)(230)$ and $s < (200)(250)$;
that is, $41{,}400 < s < 50{,}000$.

14. (a) If the side of a square is x inches long, then the side of the triangle is $x + 3.5$ inches long, and

$$4x = 3(x + 3.5).$$

The length of the side of the square is 10.5 inches.

(b) If the rate of the current is x miles per hour, then the rate of the boat downstream is $x + 10$ miles per hour, and

$$x + 10 \leq 25.$$

The rate of the current is equal to or less than 15 miles per hour.

(c) If x is the number of hours spent on the job, then

$$3 \leq x \leq 5.$$

Mary can expect to spend from 3 to 5 hours on the job.

(d) If x is the number of hours Jim must work,

$$1.5\, x \geq 75.$$

Jim must work at least 50 hours.

Chapter 8

Suggested Test Items

1. Find the truth sets of the following open sentences and draw their graphs.
   (a) $(-x) + 5 < (-8) + |-8|$
   (b) $\frac{3}{2}x + (-3) > x + (-4)$
   (c) $(-7) + (-y) < \frac{3}{7} + (-\frac{3}{7})$
   (d) $37 + (-6r) + 7 > 9r + (-7r) + 8 + (-2r)$
   (e) $5n + (-3) > 2n + 9$
   (f) $4|x + (-3)| > 12$

2. If p, q and t are real numbers and $p < q$, which of the following sentences are true?
   (a) $p + t < q + t$, if $t > 0$
   (b) $p + t > q + t$, if $t < 0$
   (c) $pt < qt$, if $t > 0$
   (d) $pt > qt$, if $t < 0$
   (e) $\frac{1}{p} < \frac{1}{q}$

3. If n is a non-negative number and x is a non-positive number, which of the following are true?
   - (a) $x \leq 0$
   - (b) $n \nless x$
   - (c) $n \geq 0$
   - (d) $x \leq n$
   - (e) $x > 0$
   - (f) $n > x$

4. We know that the sentence "$4 < 7$" is true. What true sentences result when both numbers are
   - (a) increased by 5
   - (b) decreased by 5
   - (c) multiplied by 5
   - (d) multiplied by (-5)
   - (e) multiplied by 0

5. Write an open sentence for each of the following problems. State the truth sets and answer the questions.
   - (a) Tom has \$15 more than Bill. After Tom spends \$3 for meals, the two boys together have at least \$60. How much money does Bill have?
   - (b) If 13 is added to a number and the sum is multiplied by 2, the product is more than 76. What is the number?
   - (c) Tom works at the rate of p dollars per day. After working 5 days he collects his pay and spends \$6 of it. If he then has more than \$20 left, what was his rate of pay?
   - (d) A farmer discovered that at least 70% of a certain kind of seed grew into plants. If he has 245 plants, how many seeds did he plant?

5. Which of the following sentences are true for every a and every b?
   - (a) If $a + 2 = b$, then $b < a$.
   - (b) If $a + (-3) = b$, then $b < a$.
   - (c) If $(a + 5) + (-2) = b + 5$, then $b < a$.
   - (d) If $a < 4$ and $4 > b$, then $a < b$.
   - (e) If $a + 2 < 7$ and $b + 2 > 7$, then $a < b$.

7. Given $\frac{7}{9}$, $\frac{2}{3}$, $\frac{3}{4}$ and n. In each part of this problem make as many statements involving "<" about n and the given numbers as you can, if you know:

(a) $n < \frac{7}{9}$ (b) $n < \frac{2}{3}$ (c) $n < \frac{3}{4}$

8. A man has three ore samples, each having the same volume. The sample of lead outweighs the sample of iron. The sample of gold outweighs the sample of lead. Which is a heavier ore sample, gold or iron? What property of real numbers is illustrated here?

Chapter 9

SUBTRACTION AND DIVISION FOR REAL NUMBERS

The culmination of our study of real numbers came at the end of Chapter 8. There we described the real number system as a set of elements (real numbers) for which two binary operations are given and assumed to satisfy a list of basic properties (axioms). The two operations are addition and multiplication. Thus, all of algebra could be developed without even mentioning subtraction or division. However, it is convenient to have the binary operations of subtraction and division, if only for ease in writing. Evidently, these operations must be defined directly in terms of the basic operations of addition and multiplication.

There are two equivalent ways of defining subtraction either of which could have been used here. They are

(1) $a - b = a + (-b)$

(2) $a - b$ is the solution of the equation $a = b + x$.

The writers of this book chose the first of these because it lends itself more readily to the point of view that subtraction is a kind of inverse operation to addition which is already known for numbers of arithmetic and must be extended to all real numbers. Thus we have only to identify subtraction in arithmetic with $a + (-b)$ in order to motivate the definition for all real numbers. This definition also builds on the work done previously with the additive inverse, which is important in its own right, and fits in nicely with the picture of addition and subtraction in the number line. See Theorem 9-1 in which the two definitions are proved equivalent.

There are also two ways of defining division:

(3) $\frac{a}{b} = a \cdot \frac{1}{b}$

(4) $\frac{a}{b}$ is the solution of the equation $a = bx$, $b \neq 0$.

In this case also the first method was chosen because it parallels the chosen definition of subtraction and emphasizes the multiplicative inverse. It should also be mentioned that under these definitions the various properties of subtraction and division flow easily from earlier properties of addition and multiplication.

The second method of defining subtraction and addition uses "solution of equations" as motivation. It has some advantage when the objective is to motivate extensions of the number system by demanding that certain simple equations always have solutions. For example the equation $a = b + x$ does not always have a solution in the positive integers (even if a and b are positive integers) but does always have a solution when the system is extended to include the negative integers. Similarly, the equation $a = bx$ $(b \neq 0)$ does not always have a solution in the integers (even if a and b are integers) but does always have a solution when the system is expanded to include the rational numbers. In later courses the introduction of the complex numbers is motivated by the demand that $x^2 + a = 0$ (in particular $x^2 + 1 = 0$) have a solution for every a.

Once these two ways of defining subtraction and division are shown to be equivalent (See Theorems 9-1, 9-4), we are then free to use whichever one is most appropriate to the problem at hand. You will notice that we have availed ourselves of this freedom in many places.

The student is motivated by being asked to describe subtraction of numbers of arithmetic in terms of what must be added to the smaller to obtain the larger. When it is established that we must add the opposite of the smaller, we immediately take this as the definition of subtraction for all real numbers. A similar motivation leads to the definition of division.

Reference to subtraction and division will be found in Studies in Mathematics, Volume III, pages 3.11 - 3.14.

9-1. Definition of Subtraction

We assume that the student is familiar in arithmetic with subtracting b from a by finding how much must be added to b to obtain a. From this our knowledge of equivalent equations quickly leads to adding the opposite of b to a.

[pages 209-211]

For the student who has been subtracting by "taking away", we hope the illustration of making change will help the transition to an additive viewpoint.

Page 210. $(-5) - 2 = (-5) + (-2) = -7$

$5 - (-2) = 5 + 2 = 7$

$(-2) - (-5) = (-2) + 5 = 3$

$(-5) - (-2) = (-5) + 2 = -3$

We read "5 - (-2)" as "five minus the opposite of 2". The first "-" indicates subtraction. The second "-" means "the opposite of". (Of course in this case the second could also be read "negative 2". If, a variable were involved, however, the "-" would have to be read "the opposite of".)

We shall soon want our students to be able to look at $a - b$ and think of it as a sum, the sum of a and $(-b)$. This is justified by our definition of subtraction.

You have, no doubt, noticed that we are not using the word "sign" for the symbol "-" or "+". We find that we do not really need the word, and since its misuse in the past has caused considerable lack of understanding (in such things as "getting the absolute value of a number by taking off its sign") we prefer not to use "sign" as a word.

A related point that we should mention is that we do not write + 5 for the number five. The positive numbers are the numbers of arithmetic. We therefore do not need a new symbol for them. Thus we write 5, not + 5, and the symbol "+" is used only to indicate addition.

Page 210. Theorem 9-1 proves that our definition of subtraction is consistent with the viewpoint that $a - b$ is the number which added to b gives a. From now on either viewpoint is available.

Answers to Problem Set 9-1; pages 211-212:

1. -3000
2. $\frac{5}{4}$

3. $\frac{3}{2}$

4. -1.262

5. -3.01

6. 5

7. 160

8. -2

9. $15 - (-8) = 23$

10. $(-25) - (-4) = -21$

11. $(-9) - 6 = -15$

12. $(-12) - (-17) = 5$

13. $8 - (-5) = 13$

14. S is a subset of R and is actually equal to R. The real numbers are closed under subtraction. The numbers of arithmetic are not closed under subtraction. 3-5 is not a number of arithmetic and serves as a counter example.

15. The left member

$a - a = a + (-a)$ by definition of subtraction,

$= 0$ by property of opposites for addition

16. (a) The student may apply theorem 9-1;

since $y - 725 = 25$

then $y = 725 + 25$

$y = 750$.

He may equally well apply the addition property of equality after first using the definition of subtraction;

$y - 725 = 25$

$y + (-725) = 25$,

which is equivalent to

$y = 25 + 725$

$y = 750$

The truth set is $\{750\}$.

If you have not already done so, you may by this time want to allow your students to omit writing "which is equivalent to". You should, however, check occasionally to be sure that the students continue to think clearly about

(1) performing on both members of a sentence only those operations which they are sure will produce an equivalent sentence, or

(2) merely changing the name of one member or the other of the sentence.

(b) {110}

(c) {-12}

(d) {-4}

(e) {-16.4}

(f) $\{\frac{1}{4}\}$

17. -3 - 10 = -13 ; the new temperature is 13° below zero.

18. 7.23 - 15.50 = -8.27 ; her account shows a debit of $8.27.

19. (-80) - (-50) = -30

20. 14,495 - (-282) = 14,777; the top of Mt. Whitney is 14,777 feet above the lowest part of Death Valley.

9-2. Properties of Subtraction

Page 212. Subtraction is neither commutative nor associative.

Page 213. The decision that $a - b - c$ shall mean $a + (-b) + (-c)$ is an arbitrary one. At the same time we can see that it is a convenient choice. Because of it we may think of $6a - 2x + b - 3y$ as the sum of $6a, -2x, b,$ and $-3y$. We hope the students will soon be able to think this way.

Page 214. Example 3. The answers to the three Why's are

(1) Definition of subtraction

(2) $-(ab) = (-a)b$ and $-(-a) = a$

(3) Associative and commutative properties of addition and the distributive property.

Page 215 The justification of each of the steps in the subtraction

$$(6a - 8b + c) - (4a - 2b + 7c)$$

are:

(1) Definition of subtraction.

(2) $-(x + y + z) = (-x) + (-y) + (-z)$ for all real numbers $x, y,$ and z.

(3) $-(-x) = x$ for all real numbers x.

(4) Associative and commutative properties of addition.

(5) $-xy = (-x)y$ and $x = 1 \cdot x$ for any real numbers x and y.

(6) Distributive property.

(7) Addition of real numbers.

(8) $(-x)(y) = -xy$ for any real numbers x and y.

(9) Agreement that $a - b - c$ means $a + (-b) + (-c)$.

Answers to Problem Set 9-2a; page 216:

1. (a) $-x$
   (b) $-8a$
   (c) $11x^2$
   (d) $-5xz$
2. (a) $-7y$
   (b) $\frac{3}{2}c$
   (c) $-8x^2$
   (d) $-2a^2 + 6a$
3. (a) $-3x + 4y$
   (b) $5a + 3y$
   (c) $2a$
   (d) $x-7$
4. -4
5. $\frac{1}{3}$
6. 1
7.

$(3\pi + 9) - (5\pi - 9) = (3\pi + 9) + \left(-(5\pi - 9)\right)$	definition of subtraction
$= (3\pi + 9) + \left((-5\pi) + 9\right)$	the opposite of a sum is the sum of the opposites
$= \left(3\pi + (-5\pi)\right) + (9 + 9)$	associative and commutative properties of addition
$= \left(3\pi + (-5)\pi\right) + (9 + 9)$	$-ab = (-a)b$
$= \left(3 + (-5)\right) \pi + (9 + 9)$	distributive property
$= (-2)\ \pi + 18$	addition of real numbers
$= -2\pi + 18$	$-ab = (-a)\ b$

8. $\sqrt{5} + 10$
9. $2y$
10. $-\frac{5}{2}c$
11. $1-x$
12. 0
13. $a - 11b$
14. $4x^2 + x + 15$
15. $-2a + 5b - c - 4$
16. $11x^2 - 10x - 8$
17. $3x^2 + 4x + 8$
18. $-xy + 3yz - 4xz$
19. $10n + 13p - 13a$

20. $(5x - 3y) - (2 + 5x) + (3y - 2)$

$= \big(5x + (-3y)\big) + \big(-(2 + 5x)\big) + \big(3y + (-2)\big)$ definition of subtraction

$= \big(5x + (-3y)\big) + \big((-2) + (-5x)\big) + \big(3y + (-2)\big)$ the opposite of a sum is the sum of the opposites

$= \big(5x + (-5x)\big) + \big((-3y) + 3y\big) + \big((-2) + (-2)\big)$ associative and commutative properties of addition

$= 0 + 0 + \big((-2) + (-2)\big)$ addition property of opposites

$= -4$ addition of real numbers

21. $(11a + 13b - 7c) - (8a - 5b - 4c) = 3a + 18b - 3c$

22. $(-3x + 12) - (-3x^2 + 5x - 7) = 3x^2 - 8x + 19$

23. $(-9s - 3u) - (3s - 4t + 7u) = -12s - 10u + 4t$

24. If $a > b$, then

$a + (-b) > b + (-b)$, addition property of order
$a - b > b + (-b)$, definition of subtraction
$a - b > 0$. addition property of opposites

25. If $(a - b)$ is a positive number, then $a > b$.
If $(a - b)$ is a negative number, then $a < b$.
If $(a - b)$ is zero then $a = b$.

26. If a, b, and c are real numbers, and $a > b$, then $a - c > b - c$.

If $a > b$,
$a + (-c) > b + (-c)$, addition property of order
$a - c > b - c$. definition of subtraction

Answers to Problem Set 9-2b; pages 217-218:

1. (a) -8
   (b) $2a - 8$
   (c) -27
   (d) $-5x$
   (e) $4x - 12$
   (f) $15 - 10x$
   (g) $-6x - 6$
   (h) $-6x - 6$
   (i) $ab - 2a$
   (j) $xy + 4y$

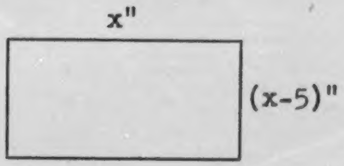
2. (a) $3a - 6b + 3c$
   (b) $-7x$
   (c) $a - 2b$
   (d) $8u^2 + 6u - 9$
   (e) $x^2 - y^2$
   (f) $2a^2 - ab - b^2$
   (g) $a - 2b + 5c$
   (h) $-6x^2 - xy$
   (i) $-3ab - ac + 3a$
   (j) $a^2 + 2ab + 6a + b^2 + 6b + 9$

3. (a) By now the students are probably writing only the following:

$$\begin{aligned} 3x - 4 &= 5 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

   The truth set is $\{3\}$.
   (b) $\{1\}$
   (c) $\{2\}$
   (d) all y such that $y > -\frac{1}{2}$
   (e) all u such that $u > -\frac{2}{3}$
   (f) $\{-\frac{16}{7}\}$
   (g) all x
   (h) all a
   (i) all c such that $c > 3$

4. (a) If the rectangle is x inches long, then it is $x - 5$ inches wide and



$$\begin{aligned} 2x + 2(x - 5) &= 38 \\ 4x - 10 &= 38 \\ x &= 12 \end{aligned}$$

   The length is 12 inches.

(b) The number is 51.

(c) The teacher has less than 23 students.

Page 217. $(-3)(2x - 5) = (-3)\big(2x + (-5)\big)$ definition of subtraction

$= (-3)(2x) + (-3)(-5)$ distributive property

$= \big((-3)(2)\big)x + 15$ associative property and definition of multiplication

$= (-6)x + 15$ definition of multiplication

$= -6x + 15$ $(-a)(b) = -(ab)$

In the abbreviated form

$$(-3)(2x - 5) = -6x + 15$$

it may appear as though we are assuming a distributive property of multiplication over subtraction. Actually, since we are thinking of $2x - 5$ as the sum of $2x$ and (-5), we are still using the distributive property of multiplication over addition and do not need to extend it to subtraction.

9-3 Subtraction in Terms of Distance

The relation between the difference of two numbers and the distance between their points on the number line is introduced here to make good use again of the number line to help illustrate our ideas.

You are no doubt aware, however, of the fact that $(a - b)$ as a directed distance and $|a - b|$ as a distance are very helpful concepts in dealing with slope and distance in analytic geometry.

Answers to Problem Set 9-3; pages 220-223:

(1) (a) $5 - (-3) = 8$
- (b) $|5 - (-3)| = 8$
- (c) $-2 - 6 = -8$
- (d) $|-2 - 6| = 8$
- (e) $1 - 5 = -4$
- (f) $|1 - 5| = 4$
- (g) $-2 -(-8) = 6$
- (h) $|-2 -(-8)| = 6$
- (i) $0 - 7 = -7$
- (j) $|0 - 7| = 7$

(2) (a) $5 - x$
- (b) $|5 - x|$
- (c) $x + 2$
- (d) $|x + 2|$
- (e) $-x + 1$
- (f) $|-1 + x|$
- (g) $x - 0 = x$
- (h) $|x - 0| = |x|$

3. (a) $|9 - 2| = |9| - |2|$
   (b) $|2 - 9| > |2| - |9|$
   (c) $|9 - (-2)| > |9| - |-2|$
   (d) $|(-2) - 9| > |-2| - |9|$
   (e) $|(-9) - 2| > |-9| - |2|$
   (f) $|2 - (-9)| > |2| - |-9|$
   (g) $|(-9) - (-2)| = |-9| - |-2|$
   (h) $|(-2) - (-9)| > |-2| - |-9|$

4. From the preceding exercise the student will, we hope, infer that for any real numbers a and b,

$$|a - b| \geq |a| - |b| ,$$
$$|a - b| \geq |b| - |a| ,$$
$$|a - b| \geq \big||a| - |b|\big| .$$

In case some of the more capable students are interested in seeing a proof of these statements, we give the following. The statement that $|x + y| \leq |x| + |y|$ for any real numbers x and y can be used to prove the three statements above (see Problem 7, Problem Set 8-2b):

With $x = a - b$ and $y = b$, we have

$$|a| = |(a - b) + b| \leq |a - b| + |b| .$$

By the addition property of order,

$$|a| + (-|b|) \leq |a - b| ,$$
$$|a| - |b| \leq |a - b| ,$$
$$|a - b| \geq |a| - |b| .$$

Similarly, $x = b - a$ and $y = a$ leads to the sentence

$$|b - a| \geq |b| - |a| .$$

Since $|b - a| = |-(b - a)| = |a - b|$, this gives

$$|a - b| \geq |b| - |a| .$$

To prove the third sentence, notice that $|b| - |a| = -(|a| - |b|)$, so that we now have

$$|a - b| \geq |a| - |b| ,$$
$$|a - b| \geq -(|a| - |b|).$$

[pages 220-221]

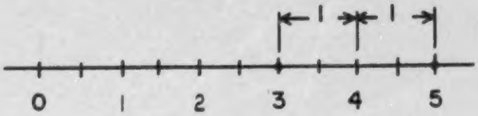
In other words, if $|a| \neq |b|$, then $|a - b|$ is greater than or equal to the larger of $|a| - |b|$ and its opposite $-(|a| - |b|)$, and this means that

$$|a - b| \geq \big||a| - |b|\big| \;;$$

if $|a| = |b|$, then $\big||a| - |b|\big| = |0| = 0$, so that the latter inequality is also true in this case.

5. The distance between a and b is found to be at least as great as the distance between $|a|$ and $|b|$, because a and b can be on opposite sides of 0, while $|a|$ and $|b|$ must be on the same side.

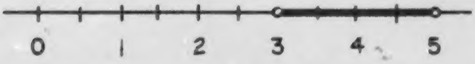
6. The two numbers x such that the distance between x and 4 is 1 are the numbers corresponding to the two points 1 unit away from 4.



Though the above is the suggested approach to this problem, some students may do it by using the definition of absolute value.

If $x - 4 \geq 0$, then $1 = |x - 4| = x - 4$ and, so, $x = 5$; if $x - 4 < 0$, then $1 = |x - 4| = -(x - 4) = -x + 4$, from which we get $x = 3$.

7. The truth set of the sentence $|x - 4| < 1$ is the set $3 < x < 5$.



Rather than using formal methods for solution of the inequality, the student will be guided by the question: What is the set of numbers x such that the distance between x and 4 is less than 1? As in the case of the preceding exercise, the student may work directly from the definition of absolute value instead of by the suggested approach.

For example, if $x - 4 \geq 0$, then $|x - 4| = x - 4$ and

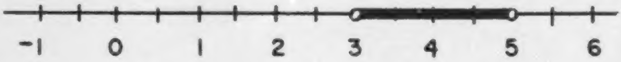
$$x - 4 < 1$$
$$x < 5 .$$

If $x - 4 < 0$, then $|x - 4| = -x + 4$ and

$$-x + 4 < 1$$
$$-x < -3$$
$$x > 3$$

8. All x such that $x < 3$ or $x > 5$

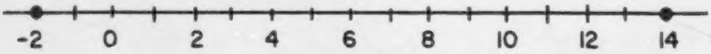
9. The graph of $x > 3$ and $x < 5$ is



It is the same as the truth set of $|x - 4| < 1$.

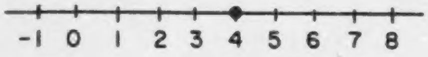
10. (a) Since $|x - 6|$ represents the distance between 6 and x, the sentence $|x - 6| = 8$ tells us that the point with coordinate x must be 8 units away from 6. Thus x must be $6 + 8$ or $6 - 8$. The truth set is $\{-2, 14\}$.

Graph:



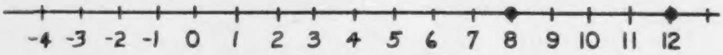
(b) Truth set: $\{4\}$

Graph:



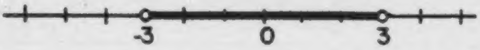
(c) Truth set: $\{8, 12\}$

Graph:



(d) Truth set: Real numbers x such that $-3 < x < 3$.

Graph:



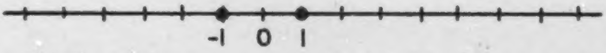
(e) Truth set: All real numbers.

Graph: Real number line.

(f) $|y| = 1$

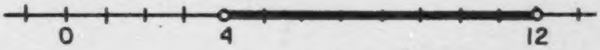
Truth set: $\{-1, 1\}$

Graph:



(g) Truth set: Real numbers y such that $4 < y < 12$.

Graph:

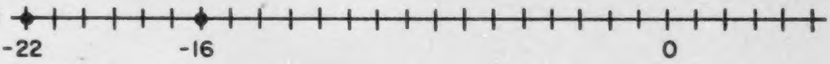


(h) $|z| = -6$.

Truth set: The empty set $\emptyset$.

(i) Truth set: $\{-22, -16\}$

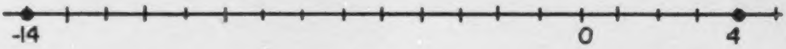
Graph:



(j) $|y + 5| = |y - (-5)| = 9$

Truth set: $\{-14, 4\}$

Graph:



11. Think of $|x|$ as the distance on the number line between 0 and x.

$|x| = 3,\quad x = -3$ or $x = 3$

$|x| < 3,\quad x > -3$ and $x < 3$

$|x| \leq 3,\quad$ None

$|x| > 3,\quad x < -3$ or $x > 3$

$|x| \geq 3,\quad x \leq -3$ or $x \geq 3$

$|x| \nless 3,\quad x \leq -3$ or $x \geq 3$

$|x| \ngtr 3,\quad$ None

*12. Let the distance in miles to the east of the 0 mark correspond to positive numbers:

		John's position on the number line	Rudy's position on the number line	The difference	Distance between them in miles
(a)	(1)	$(3)(10) = 30$	$(3)(-12) = -36$	$\lvert 30-(-36)\rvert = 66$	66
	(2)	$(1\frac{1}{2})(10) = 15$	$(1\frac{1}{2})(-12) = -18$	$\lvert 15-(-18)\rvert = 33$	33
	(3)	$(\frac{1}{3})(10) = \frac{10}{3}$	$(\frac{1}{3})(-12) = -\frac{12}{3}$	$\lvert \frac{10}{3}-(-\frac{12}{3})\rvert = \frac{22}{3}$	$\frac{22}{3}$
(b)	(1)	$5 + 3(10) =$	$-6 + 3(12) =$	$\lvert 35 - 30\rvert = 5$	5
		$5 + 30 = 35$	$-6 + 36 = 30$		
	(2)	$5 + (1\frac{1}{2})(10) =$	$-6 + (1\frac{1}{2})(12) =$	$\lvert 20 - 12\rvert = 8$	8
		$5 + 15 = 20$	$-6 + 18 = 12$		
	(3)	$5 + (\frac{1}{3})(10) =$	$-6 + (\frac{1}{3})(12)$	$\lvert \frac{25}{3} - (-2)\rvert =$	$\frac{31}{3}$
		$5 + \frac{10}{3} = \frac{25}{3}$	$-6 + 4 = -2$	$\lvert \frac{25}{3} + \frac{6}{3}\rvert = \frac{31}{3}$	
(c)	(1)	$(3)(10) = 30$	$(2\frac{3}{4})(-12) =$	$\lvert 30-(-33)\rvert = 63$	63
			$(\frac{11}{4})(-12) = -33$		
	(2)	$(1\frac{1}{2})(10) = 15$	$(1\frac{1}{4})(-12) =$	$\lvert 15-(-15)\rvert = 30$	30
			$(\frac{5}{4})(-12) = -15$		
	(3)	$(\frac{1}{3})(10) = \frac{10}{3}$	$(\frac{1}{3} - \frac{1}{4})(-12) =$	$\lvert \frac{10}{3} -(-1)\rvert =$	$\frac{13}{3}$
			$(\frac{1}{12})(-12) = -1$	$\lvert \frac{10}{3} + 1\rvert = \frac{13}{3}$	
(d)	(1)	$(3)(-10) = -30$	$-6 + 3(-12) = -42$	$\lvert (-30)-(-42)\rvert = 12$	12
	(2)	$(1\frac{1}{2})(-10) = -15$	$-6 + (1\frac{1}{2})(-12) = -24$	$\lvert (-15)-(-24)\rvert = 9$	9
	(3)	$(\frac{1}{3})(-10) = -\frac{10}{3}$	$-6 + (\frac{1}{3})(-12) = -\frac{30}{3}$	$\lvert (-\frac{10}{3})-(-\frac{30}{3})\rvert = \frac{20}{3}$	$\frac{20}{3}$

9-4. Division

See the notes at the beginning of this chapter about our choice of definition of division.

Page 223. You will notice that we are becoming more relaxed about our use of words. We shall feel free to use "numerator" for either the upper numeral in a fraction or the number which that numeral represents. This is the way words are used commonly in mathematics, so we might as well start getting used to it. In most situations there is no confusion. When necessary we can go back to the original precise meaning, and it is for this purpose that we should take care to establish the precise meaning before we start to relax, and we should recall that precise meaning from time to time:

If you find that your students have had earlier viewpoints of division other than "what times 2 gives 10?", it should still be not difficult to satisfy them by examples that our definition for real numbers agrees with their experience in arithmetic.

Answers to Problem Set 9-4a; page 224:

1. 15
2. 1
3. 2500
4. 6
5. -6
6. -6
7. 4
8. $\frac{20}{3}$
9. $\frac{7}{16}$
10. 2
11. 2
12. $\frac{x}{y}$

Page 225. It may help to recall this relation between division and multiplication from the earlier experience of the students if you ask them what they have learned to do to check division. This is an example of using the "only if" part of Theorem 9-4, that is, for $b \neq 0$, if $a = cb$, then $\frac{a}{b} = c$.

It is this theorem which shows the equivalence between our definition of division $(\frac{a}{b} = a \cdot \frac{1}{b})$ and the other common definition $(\frac{a}{b}$ is c such that $a = cb)$.

Page 225. We have tried to give a hint of how one knows where to start in a proof of this sort. Try to sieze opportunities to give such hints to your students wherever possible. There is, of course, not a single pattern for all proofs, and we do not expect to make expert provers of our ninth grade students. It is, nevertheless, by many experiences with observing patterns in proofs that understanding and ability are developed.

The reasons for the steps in the proof of Theorem 9-4 are as follows:

First part, (1) Definition of division.
(2) Multiplication property of equality.
(3) Associative property of multiplication.
(4) Multiplication property of reciprocals.
(5) Multiplication property of 1.

Second part, (1) Multiplication property of equality.
(2) Associative property of multiplication.
(3) Multiplication property of reciprocals.
(4) Multiplication property of 1.
(5) Definition of division.

Answers to Problem Set 9-4b; pages 226-229:

1. To prove: $\frac{a}{1} = a$

Proof: $a = a(1)$ property of one for multiplication
$\frac{a}{1} = a$ Theorem 9-4 with $c = a$, $b = 1$

Notice that the above proof begins with a true sentence and an equivalent sentence is written by applying an "if and only if" theorem.

2. To prove: $\frac{a}{a} = 1$

Proof: $a = a(1)$ property of one for multiplication

$a = (1)a$ commutative property for multiplication

$\frac{a}{a} = 1$ Theorem 9-4 with $c = 1$ and $b = a$

3. (a) -1250, $(-1250)(-2) = 2500$

(b) -9, $(-9)(5) = -45$

(c) 4, $(4)(-50) = -200$

(d) $\sqrt{5}$, $(\sqrt{5})(3) = 3\sqrt{5}$

(e) 5, $(5)(7p) = 35p$

(f) 3π, $(3\pi)(3) = 9\pi$

(g) $-\frac{2}{9}$, $(-\frac{2}{9})(3) = -\frac{2}{3}$

(h) 36, $(36)(\frac{1}{3}) = 12$

(i) $\frac{5}{7}$, $(\frac{5}{7})(-\frac{7}{8}) = -\frac{5}{8}$

(j) 1, $(1)(-976) = -976$

(k) 0, $(0)(48) = 0$

(ℓ) $\frac{180}{\pi}$, $(\frac{180}{\pi})(2\pi) = 360$

(m) $-5a$, $(-5a)(-3) = 15a$

(n) -140, $(-140)(0.1) = -14$

(o) $-m$, $(-m)(-93) = 93m$

(p) 7, $(7)(2\sqrt{2}) = 14\sqrt{2}$

4. If we assume the definition of division holds when $b = 0$, then $\frac{28}{0} = 28\cdot\frac{1}{0}$. But zero has no reciprocal.

We can also look at $\frac{28}{0}$ from the standpoint of Theorem 9-4, assuming the theorem holds when $b = 0$. If $\frac{28}{0} = c$ then $28 = c\cdot 0$. But $c\cdot 0$ is zero and cannot be a name for 28.

5. A positive number divided by a negative number is negative. We can see this by looking at the definition of division for $a > 0$ and $b < 0$.

$$\frac{a}{b} = a\left(\frac{1}{b}\right)$$

If $b < 0$ then $\frac{1}{b} < 0$ by a theorem of Chapter 7 which says, "The reciprocal of a negative number is negative". Thus, $a\left(\frac{1}{b}\right)$ is the product of a positive number and a negative number, and is negative.

By similar reasoning we can say: A negative number divided by a positive number is negative and a negative number divided by a negative number is positive.

6. (a) Help the students to see that the truth sets of these sentences can be found in more than one way.

   (1) They can apply Theorem 9-4.
   If $6y = 42$, then $y = \frac{42}{6}$.

   (2) They can get an equivalent sentence by using the multiplication property of equality.

$$6y = 42$$
$$y = \frac{1}{6}\cdot 42$$
$$y = 7$$

   The truth set is $\{7\}$.

(b) $\{-7\}$

(c) $\{-7\}$

(d) $\{7\}$

(e) $\{\frac{1}{7}\}$

(f) $\{1\}$

(g) $\{\frac{43}{6}\}$

(h) $\{100\}$

(i) $\{75\}$

(j) $\{12\}$

(k) $\{0\}$

(ℓ) $\{\frac{2}{3}\}$

7. (a) $5a - 8 = -53$
$5a = -45$
$a = -9$

The truth set is $\{-9\}$.

(b) $\{16\}$

(c) $\{5\}$

(d) $\{28\}$

(e) $\{18\}$

8. If n is the number, then

$$6n - 5 = -37 ,$$
$$6n = -32 ,$$
$$n = -\frac{32}{6} \text{ or } -\frac{16}{3} .$$

Hence, the number is $-\frac{16}{3}$.

9. If s is the number, then

$$32 + \frac{2}{3}s = 38 ,$$
$$\frac{2}{3}s = 6 ,$$
$$s = 6\cdot\frac{3}{2} \text{ or } 9 .$$

Hence, the number is 9.

10. $\frac{70}{3}$ pound of sugar are needed for 35 cakes.

11. The rectangle is 9 inches wide.

12. Dick is 7 years old and John is 21 years old.

13. 22, 24

14. The sum of any two consecutive numbers from the set of odd integers between 0 and 42 would satisfy the conditions of the problem.

15. If the price was p dollars before the sale,

$$p - .20p = 30 ,$$
$$.80p = 30 ,$$
$$p = 30\cdot\frac{100}{80} \text{ or } 37.50 .$$

Hence, the original price was \$37.50.

16. $1\frac{1}{4}$ hours

17. The number is 9.

18. Let s be the number of 4¢ stamps bought. If she was charged the correct amount, then s must be a non-negative integer and

$$\begin{aligned} 15(.03) + .04s &= 1.80 \ , \\ .45 + .04s &= 1.80 \ , \\ .04s &= 1.35 \ , \\ 4s &= 135 \ , \\ s &= \frac{135}{4} = 33\frac{3}{4} \ . \end{aligned}$$

Since $33\frac{3}{4}$ is not an integer, she was charged the wrong amount.

19. He has 12 pennies, 16 dimes, 22 nickels. He has $2.82.

20. John has $25.

21. 20 miles per hour.

22. 35, 36, and 37.
Notice in this one that one may represent the three integers by i, $i + 1$, and $i + 2$. Another choice, though, which is a little simpler, is $i - 1$, i, and $i + 1$.

23. If n is the first positive integer, then

$n + 1$ is the next integer, and

$$\begin{aligned} n + (n + 1) &< 25 \ , \\ 2n + 1 &< 25 \ , \\ 2n &< 24 \ , \\ n &< 12 \ . \end{aligned}$$

Hence, the integers are (1, 2) , (2, 3) ,..., (11, 12)

*24. If x is the number of gallons of maple syrup then $160 - x$ is the number of gallons of corn syrup and

$$8x + 2.4\,(160 - x) = 608.$$

The mixture consists of 40 gallons of maple syrup and 120 gallons of corn syrup.

25. If $\frac{a}{b} > o$ and $b \neq o$, then $ab > o$.

Proof:

$\frac{a}{b} > o$ and $b \neq o$	is true by hypothesis
$\frac{a}{b} = p$ where $p > o$	let p name the same number as $\frac{a}{b}$.
$a = pb$	Theorem 9-4
$ab = pb^2$	multiplication property of equality
$b^2 > o$	Theorem 8-3b
$pb^2 > o$	The product of two positive numbers is positive.
$ab > o$	ab and pb^2 are names for the same number.

If $\frac{a}{b} < o$ and $b \neq o$, then $ab < o$.

Proof:

$\frac{a}{b} < o$ and $b \neq o$	is true by hypothesis
$\frac{a}{b} = n$ where $n < o$	Let n name the same number as $\frac{a}{b}$.
$a = bn$	Theorem 9-4
$ab = b^2n$	multiplication property of equality
$b^2 > o$	Theorem 8-3b
$b^2n < o$	The product of a positive number and a negative number is negative.
$ab < o$	ab and pb^2 are names for the same number.

9-5. Common Names

In this and the following section we are interested in four commonly accepted conventions about the simplest numeral for a number.

(1) There should be no indicated operations remaining which can be performed.

(2) If there is an indicated division, the numbers whose division is indicated should have no common factor.

(3) We prefer $-\frac{a}{b}$ to $\frac{-a}{b}$ or $\frac{a}{-b}$.

(4) It is always possible to avoid more than one indicated division.

Thus, to illustrate the first convention we would say that "$\frac{20}{4}$" is not as simple as "5"; "$\frac{2+3}{6}$" is not as simple as "$\frac{5}{6}$"; "$\frac{3\cdot 5}{7}$" is not as simple as "$\frac{15}{7}$"; but $\frac{x+3}{y}$ cannot be simplified. Similarly, for the second convention "$\frac{14}{21}$" is not as simple as "$\frac{2}{3}$" and "$\frac{2x^2+4}{ax^2+2a}$" is not as simple as "$\frac{2}{a}$" . Simplifications of this kind depend on Theorem 9-5, the theorem $\frac{a}{a} = 1$ for $a \neq 0$ and the property of 1.

Page 230. The reasons in the proof of Theorem 9-5 are:

(1) Definition of division.
(2) Associative and commutative properties of multiplication.
(3) Theorem 7-8d.
(4) Definition of division.

Answers to Problem Set 9-5a; pages 232-233:

1. If a and b are real numbers, $b \neq 0$, then $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$

Proof: $\frac{-a}{b}\cdot 1 = \frac{-a}{b}$ is true by $a\cdot 1 = a$

$\frac{-a}{b}\cdot\frac{-1}{-1} = \frac{-a}{b}$ $1 = \frac{a}{a}$, $a \neq 0$,

$\frac{(-a)(-1)}{(b)(-1)} = \frac{-a}{b}$ Theorem 9-5

$\frac{a}{-b} = \frac{-a}{b}$ $(-1)a = -a$

Also $\frac{-a}{b} = (-a)(\frac{1}{b})$ is true by the definition of division

$\frac{-a}{b} = -(a \cdot \frac{1}{b})$ $(-a)(b) = -(ab)$

$\frac{-a}{b} = -\frac{a}{b}$ definition of division

2. (a) $\frac{4}{3}$ (b) $-\frac{4}{3}$ (c) $-\frac{4}{3}$ (d) $\frac{4}{3}$

3. (a) n (b) $-n$ (c) $\frac{1}{n}$

4. (a) $\frac{2}{3}$ (b) $\frac{2}{3}$ (c) $-\frac{2}{3}$ (d) $-\frac{2}{3}$ (e) $-\frac{2}{3}$

5. (a) y (b) y (c) $x + 1$ (d) 1

6. (a) 2 (b) -2 (c) -2 (d) -2

7. (a) $\frac{x + 2}{3}$ (b) $\frac{2x - 3}{2y - 3}$ (c) $\frac{2x + 1}{3}$ (d) $x + 2$

8. (a) $2 - a$ (b) $a - 2$ (c) $-a$ (d) -1

9. (a) $\frac{1}{t - 2}$ (b) $\frac{1}{2t - 1}$ (c) $t - 2$

10. (a) $6ab$ (b) $2a^2$ (c) $\frac{3a}{b}$ (d) $-\frac{3a}{5c}$

11. (a) $x - 1$ (b) $\frac{x + 1}{4}$ (c) $\frac{3}{4}(x - 1)$ (d) $x - 1$

12. (a) $-\frac{1}{2}$ (b) $\frac{1 - 3x - 2y}{4y - 2 - 6x}$

9-6. Fractions

The main point of this section is to develop skill in simplifying an expression to one in which there is at most one indicated division. This essentially means that we are learning to multiply, divide, and add fractions.

Page 233. We are again relaxing our rigor in the use of words. We shall allow ourselves to use "fraction" for either the symbol or the number, even though correctly speaking it means the symbol.

Thus in the preceding paragraph a precise statement would have said "to multiply, divide, and add numbers which are represented by fractions."

Now that we have begun to relax our precision of language, we shall hereafter, without further comment, feel free to use convenience of language even when it violates precision of language about numbers and numerals, as long as we are sure that the precise meaning will be understood.

We mention the word "ratio" as part of the language in certain applications There are a few problems using ratio in the following pages. You may wish to mention to the students that an equation such as $\frac{x}{1197} = \frac{z}{19}$, which equates two ratios, is often called a "proportion". It seems undesirable at present to digress into a lengthy treatment of ratio and proportion since it is just a matter of special names for familiar concepts.

Answers to Problem Set 9-6a; pages 234-235:

1. (a) $\frac{21}{16}$ (b) $-\frac{21}{16}$ (c) $\frac{21}{16}$ (d) $\frac{21}{16}$
2. (a) $\frac{6}{5}$ (b) $\frac{6}{5}$ (c) $\frac{12}{7}$ (d) $\frac{6}{5}$
3. (a) $-\frac{10}{9}$ (b) $-\frac{10}{9}$ (c) $\frac{10}{9}$ (d) $\frac{10}{9}$
4. (a) $\frac{1}{n^2}$ (b) 1 (c) $\frac{1}{nx}$ (d) $\frac{1}{2n}$
5. (a) $\frac{x^2}{12}$ (b) $\frac{x^2}{12}$ (c) $\frac{3}{4}$ (d) $-\frac{x^2}{12}$
6. (a) 5 (b) $\frac{29}{6}$ (c) 5 (d) $-\frac{29}{6}$
7. (a) $\frac{4}{3}a^3$ (b) $12a$ (c) $\frac{3m}{a}$ (d) $\frac{a^2}{9}$
8. (a) $\frac{x + 2}{4}$ (b) $\frac{3}{4}(x + 2)$ (c) $\frac{3}{4}(x + 2)$
9. (a) $\frac{(n + 3)(n + 2)}{6}$ (b) 1 (c) $\frac{2(n + 3)}{3(n + 2)}$
10. (a) y^2 (b) $2a$

11. Every rational number can be represented by a fraction. Not every fraction represents a rational number. $\frac{\pi}{3}$ is a fraction but is not a rational number.

12. If there are f faculty members,

$$\frac{f}{1197} = \frac{2}{19}$$

$$f = \frac{2}{19}\cdot 1197$$

$$f = 126$$

There are 126 faculty members.

13. If the other fund received x dollars,

$$\frac{x}{387} = \frac{2}{3}$$

$$x = \frac{2}{3}\cdot 387$$

$$x = 258$$

The other fund received $258.

Page 236. The amount of telescoping which the student will do in writing his work on adding fractions should vary from student to student and from time to time for the same student. Before long, most students should be able to write

$$\frac{x}{3} + \frac{y}{5} = \frac{x}{3}\cdot\frac{5}{5} + \frac{y}{5}\cdot\frac{3}{3} = \frac{5x + 3y}{15}$$

They should be encouraged to continue to show the multiplication by 1 in the form $\frac{5}{5}$, $\frac{3}{3}$, etc.

Answers to Problem Set 9-6b; pages 236-239:

1. (a) $\frac{11}{9}$ (b) $-\frac{1}{9}$ (c) $\frac{1}{9}$
2. (a) $\frac{7}{12}$ (b) $\frac{5}{6}$ (c) $\frac{13}{12}$
3. (a) $\frac{9}{a}$ (b) $\frac{13}{2a}$ (c) $\frac{4a + 5}{a^2}$

4. (a) $\frac{3x}{4}$ (b) $\frac{x^2}{8}$ (c) $-\frac{x}{4}$ (d) $\frac{x}{4}$

5. (a) $\frac{19a}{35}$ (b) $\frac{20 - a}{35}$ (c) $\frac{20a - 1}{35}$

6. (a) $\frac{3(x - 2)}{5}$ (b) $\frac{3}{5}x$ (c) $\frac{2x + 8}{x}$

7. If a, b, and c are real numbers and $c \neq 0$, then

$$\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}.$$

The left member is

$$\frac{a}{c} + \frac{b}{c} = a \cdot \frac{1}{c} + b \cdot \frac{1}{c} \qquad \text{definition of division}$$

$$= (a + b)\frac{1}{c} \qquad \text{distributive property}$$

$$= \frac{a + b}{c} \qquad \text{definition of division}$$

*8. If a, b, and c are real numbers, $c \neq 0$ and $d \neq 0$, then

$$\frac{a}{c} + \frac{b}{d} = \frac{ad + bc}{cd}$$

The left member is

$$\frac{a}{c} + \frac{b}{d} = \frac{a}{c} \cdot \frac{d}{d} + \frac{b}{d} \cdot \frac{c}{c} \qquad x \cdot 1 = x \text{ and } 1 = \frac{x}{x}$$

$$= \frac{ad}{cd} + \frac{bc}{dc} \qquad \frac{x}{y} \cdot \frac{p}{q} = \frac{xp}{yq}$$

$$= \frac{ad + bc}{cd} \qquad \text{commutative property of multiplication and problem 7}$$

It would be undesirable for the student to memorize this result and use it as a formula. It is given here in a problem as something interesting to prove. In simplifying sums of fractions the students should continue to use the property of 1 to make the denominators alike.

9. We suggest here by an example the procedure of "clearing of fractions" at the start of solving an equation containing fractions. While we do not insist that the student use this approach, we hope that he will, with your help, see the efficiency of "clearing of fractions" and will learn to use this method effectively.

(a)	$\{12\}$	(f)	$\{\frac{20}{9}\}$
(b)	$\{2\}$	(g)	$\{5, -5\}$
(c)	$\{2\}$	(h)	$1 < x < 5$
(d)	$\{18\}$		
(e)	$\{20\}$		

10. If one of the numbers is n, the other number is $\frac{3}{5}n$, and

$$\begin{aligned} n + \tfrac{3}{5}n &= 240 \\ 5n + 3n &= 1200 \\ 8n &= 1200 \\ n &= 150 \\ \tfrac{3}{5}n &= 90 \end{aligned}$$

Or,

If one number is n, the other number is $240-n$, and

$$\begin{aligned} n &= \tfrac{3}{5}(240-n) \\ 5n &= 720-3n \\ 8n &= 720 \\ n &= 90 \\ 240-n &= 150 \end{aligned}$$

The numbers are 150 and 90.

11. The numerator was increased by 5.

12. The number is 312.

13. Joe is 12 years old and his father is 36 years old.

14. The two numbers are 5 and 2. The sum of their reciprocals is $\frac{7}{10}$. The difference of their reciprocals is $\frac{3}{10}$.

15. The ratio is $\frac{1}{19}$.

16. (a) Joe will paint $\frac{1}{7}$ of the house in one day. Joe will paint $d \cdot \frac{1}{7}$ or $\frac{d}{7}$ of the house in d days.

    (b) Bob will paint $\frac{1}{8}$ of the house in one day and $\frac{d}{8}$ in d days.

    (c) Together Bob and Joe would paint $\frac{1}{7} + \frac{1}{8}$ of the house in one day or $\frac{d}{7} + \frac{d}{8}$ in d days.

    (d) The part of the house Joe painted plus the part of the house Bob painted equals the whole house. d represents the number of days it took Joe and Bob together to paint the house and is equal to $\frac{56}{15}$ or $3\frac{11}{15}$ days.

    (e) Together Joe and Bob will paint $\frac{1}{7} + \frac{1}{8}$ or $\frac{15}{56}$ of the house in one day.

*17. If x is the number of games the ball team must win then

$$\frac{48 + x}{154} \geq 0.6$$

$$x \geq 44.4$$

Since x must be an integer, the ball team must win at least 45 of the remaining 54 games to finish with a standing of at least .600.

The highest standing they can get is

$$\frac{48 + 54}{154} = .662$$

The lowest standing they can get is

$$\frac{48 + 0}{154} = .312$$

Page 239. Three methods of simplifying "complex fractions" are given. Each method has its advantages for certain types of problems. The student should be encouraged to use some judgment in selecting his method. For instance, when there is a single fraction in the numerator and in the denominator, Method 3 is usually most simple. When there is a sum of fractions in both the numerator and the denominator, Method 1 is effective. So in Problem Set 9-6c we might find the student using Method 1 on Problems 4, 5, 6, 7, 8, 12; Method 2 or 3 on Problems 1, 2, 3, 9, 11.

Answers to Problem Set 9-6c; pages 240-241:

1. $\frac{3}{2}$
2. $2a$
3. $\frac{a^3}{3y^3}$
4. 1
5. $\frac{48}{5}$
6. $-\frac{1}{48}$
7. 7
8. $\frac{a+1}{2(a-1)}$
9. -2
10. $-\frac{(a-b)^2}{8}$
11. $\frac{(x+8)(x+2)}{27}$
12. $\frac{y}{3}$

Answers to Review Problems

1. (a) $\frac{3}{8}$

   (b) This is not a number, since $\frac{1}{0}$ is not a number.

   (c) 0

   (d) This is not a number; if it were, we could write the number as $0 \cdot \frac{1}{0}$, but $\frac{1}{0}$ is not a number.

   (e) 0

   (f) This is not a number since $5 \div 0$ means $5 \cdot \frac{1}{0}$, and $\frac{1}{0}$ is not a number.

   (g) This is not a number, since $\frac{1}{0}$ is not a number.

   (h) -36

   (i) 3

2. (a) $(5 - 5)\cdot 7 = 0$ (d) $(7 - 6) -2 = -1$
   (b) $5 - (5\cdot 7) = -30$ (e) $3\cdot(2 - 2)\cdot 5 = 0$
   (c) $7 - (6 - 2) = 3$ (f) $(3\cdot 2 - 2)\cdot 5 = 20$

3. (a) $(-1)^2 - 4(2)(5) = -39$
   (b) $(6)^2 - 4(5)(-3) = 96$
   (c) $(-3)^2 - 4(1)(-2) = 17$
   (d) $(2)^2 - 4(1654)(0) = 4$
   (e) $(0)^2 - 4(5)(-5) = 100$
   (f) $(\frac{1}{4})^2 - 4(\frac{1}{3})(-\frac{1}{5}) = \frac{79}{240}$

4. $\frac{7}{2}$ since if x is $\frac{7}{2}$, $\frac{3x + 5}{2x - 7} = \frac{3(\frac{7}{2}) + 5}{2(\frac{7}{2}) - 7} = \frac{3(\frac{7}{2}) + 5}{0}$

5. (a) $-6x - 3$ (d) $-3x + 2y$
   (b) $a^2b^2 - ab^3$ (e) $5a^2 + 10ab - 15ac$
   (c) $m^3 + m^2$ (f) $7x^3 - 21x^2$

   (g) $$\begin{aligned}(2x - 3y)(x + 4y) &= 2x(x + 4y) - 3y(x + 4y)\\ &= 2x^2 + 8xy - 3xy - 12y^2\\ &= 2x^2 + 5xy - 12y^2\end{aligned}$$

   (h) $$\begin{aligned}(2a - 3b)(2a - 3b) &= 2a(2a - 3b) - 3b(2a - 3b)\\ &= 4a^2 - 6ab - 6ab + 9b^2\\ &= 4a^2 - 12ab + 9b^2\end{aligned}$$

6. (a) For all a, $a < 7$
   (b) $\{-4\}$
   (c) For all m, $m \geq \frac{45}{2}$
   (d) For all x, $x < 9$
   (e) $\{12\}$
   (f) For all x, $x \geq 2$ or $x \leq -2$

7. If the number is n, then

$$\frac{1}{4}n + \frac{1}{8}n < n - 25,$$

$$2n + n < 8n - 200,$$

$$200 < 5n\ ,$$

$$40 < n\ .$$

Hence, the number is greater tnan 40.

8. $\frac{1}{4}\ (\frac{x+3}{x} + \frac{x-3}{x} + \frac{x+k}{x} + \frac{x-k}{x}) = \frac{1}{4}\ (\frac{4x}{x}) = 1$

9. $\frac{3}{8}\cdot\frac{5}{5} = \frac{15}{40}$, and $\frac{9}{20}\cdot\frac{2}{2} = \frac{18}{40}$.

$\frac{15}{40} < \frac{18}{40}$; thus $\frac{3}{8} < \frac{9}{20}$ is true.

$\frac{9}{20}\cdot\frac{3}{3} = \frac{27}{60}$ and $\frac{7}{15}\cdot\frac{4}{4} = \frac{28}{60}$.

$\frac{27}{60} < \frac{28}{60}$; thus $\frac{9}{20} < \frac{7}{15}$ is true.

Then, by the transitive property, $\frac{3}{8} < \frac{7}{15}$ is true.

*10. If the first shirt cost x dollars, then

$$x - .25x = 3.75,$$

$$.75x = 3.75,$$

$$x = 5.$$

Hence, the first shirt cost \$5.00. Since he sold it for \$3.75, he lost \$1.25 on it.

If the second shirt cost y dollars, then

$$y + .25y = 3.75,$$

$$1.25y = 3.75,$$

$$y = 3.$$

Hence, the second shirt cost \$3.00. Since he sold it for \$3.75, he gained \$.75 on it.

$$(-1.25) + .75 = -.50.$$

Thus, he lost \$.50 on the two sales.

11. $x = \frac{1}{a}(y - b)$

12. Last year's cost was d dollars per dozen.

This year's cost is d dollars + c cents per dozen.

$\frac{1}{2}$ dozen will cost $\frac{d \text{ dollars} + c \text{ cents}}{2}$

$$= \frac{100d + c}{2} \text{ cents.}$$

13. (a) $(x - 1)(x + 2) = 0$

is equivalent to

$x - 1 = 0$ or $x + 2 = 0$

which is equivalent to

$x = 1$ or $x = -2$

The truth set is $\{1, -2\}$.

(b) $\{-5, -7\}$

(c) $\{0, 2\}$

(d) $\{\frac{1}{2}, 2\}$

(e) $\emptyset$

(f) $\{\frac{5}{2}\}$

14. (a) For real numbers a, b, c, d, where $b \neq 0$, $d \neq 0$,

if $\frac{a}{b} = \frac{c}{d}$.

then $\frac{a}{b}(bd) = \frac{c}{d}(bd)$ multiplication property of equality.

$(a \cdot \frac{1}{b})(bd) = (c \cdot \frac{1}{d})(bd)$ definition of division.

$(ad)(b \cdot \frac{1}{b}) = (bc)(d \cdot \frac{1}{d})$ associative and commutative properties of multiplication.

$(ad) \cdot 1 = (bc) \cdot 1$ multiplication property of reciprocals.

$ad = bc$ multiplication property of 1.

(b) For real numbers a, b, c, d, where $b \neq 0$, $d \neq 0$, $c \neq 0$,

if $\frac{a}{b} = \frac{c}{d}$

then $\frac{a}{b} \cdot \frac{b}{c} = \frac{c}{d} \cdot \frac{b}{c}$ multiplication property of equality.

[pages 243-244]

$\frac{ab}{bc} = \frac{cb}{dc}$ Theorem 9-5.

$\frac{ab}{cb} = \frac{bc}{dc}$ commutative property of multiplication.

$\frac{a}{c}\cdot\frac{b}{b} = \frac{b}{d}\cdot\frac{c}{c}$ Theorem 9-5

$\frac{a}{c}\cdot 1 = \frac{b}{d}\cdot 1$ $\frac{x}{x} = 1$ if $x \neq 0$.

$\frac{a}{c} = \frac{b}{d}$ multiplication property of 1.

(c) For real numbers a, b, c, d, where $a \neq 0$, $b \neq 0$, $c \neq 0$, $d \neq 0$,

if $\frac{a}{b} = \frac{c}{d}$

then $\frac{a}{b}\cdot\frac{bd}{ac} = \frac{c}{d}\cdot\frac{bd}{ac}$ multiplication property of equality.

$\frac{a(bd)}{b(ac)} = \frac{c(bd)}{d(ac)}$ Theorem 9-5.

$\frac{d(ab)}{c(ab)} = \frac{b(cd)}{a(cd)}$ associative and commutative properties of multiplication.

$\frac{d}{c}\cdot\frac{ab}{ab} = \frac{b}{a}\cdot\frac{cd}{cd}$ Theorem 9-5

$\frac{d}{c}\cdot 1 = \frac{b}{a}\cdot 1$ $\frac{x}{x} = 1$ if $x \neq 0$.

$\frac{d}{c} = \frac{b}{a}$ multiplication property of 1.

$\frac{b}{a} = \frac{d}{c}$ If $x = y$, then $y = x$.

(d) For real numbers a, b, c, d, where $b \neq 0$, $d \neq 0$,

if $\frac{a}{b} = \frac{c}{d}$

then $\frac{a}{b} + 1 = \frac{c}{d} + 1$ addition property of equality.

$\frac{a}{b} + \frac{b}{b} = \frac{c}{d} + \frac{d}{d}$ $\frac{x}{x} = 1$ if $x \neq 0$.

$a\cdot\frac{1}{b} + b\cdot\frac{1}{b} = c\cdot\frac{1}{d} + d\cdot\frac{1}{d}$ definition of division.

$(a + b)\frac{1}{b} = (c + d)\frac{1}{d}$ distributive property.

$\frac{a + b}{b} = \frac{c + d}{d}$ definition of division.

15. $\frac{ac}{bc} = \frac{a}{b}\cdot\frac{c}{c}$ Theorem 9-5.

$= \frac{a}{b}\cdot 1$ $\frac{x}{x} = 1$ if $x \neq 0$.

$= \frac{a}{b}$ multiplication property of 1

16. To show that the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$ we need to show that $\frac{a}{b}\cdot\frac{b}{a} = 1$.

$\frac{a}{b}\cdot\frac{b}{a} = \frac{ab}{ba}$ Theorem 9-5.

$= \frac{ab}{ab}$ commutative property of multiplication.

$= 1$ $\frac{x}{x} = 1$ if $x \neq 0$.

Hence $\frac{b}{a}$ _is_ the reciprocal of $\frac{a}{b}$ or $\frac{1}{\frac{b}{a}} = \frac{a}{b}$.

*17. (a) Yes, because the product of any two members of the set is a member of the set.

(b) $(-1)\times j = -j$; $j\times(-1) = -j$. Hence, $(-1)\times j = j\times(-1)$.
$j\times(-j) = 1$; $(-j)\times j = 1$. Hence, $j\times(-j) = (-j)\times j$.
$(-1)\times(-j) = j$; $(-j)\times(-1) = j$. Hence, $(-1)\times(-j) = (-j)\times(-1)$

(c) $\big((-1) \times j\big) \times (-j) = (-j) \times (-j) = -1$.
$(-1) \times \big(j \times (-j)\big) = (-1) \times 1 = -1$.
Hence, $\big((-1) \times j\big) \times (-j) = (-1) \times \big(j \times (-j)\big)$.
$\big(1 \times (-1)\big) \times j = (-1) \times j = -j$.
$1 \times \big((-1) \times j\big) = 1 \times (-j) = -j$.
Hence, $\big(1 \times (-1)\big) \times j = 1 \times \big((-1) \times j\big)$.

(d) Yes. $1 \times 1 = 1$.
$(-1) \times 1 = -1$.
$j \times 1 = j$.
$(-j) \times 1 = -j$.

(e) $1 \times 1 = 1$. Hence, 1 is the reciprocal of 1.
$(-1) \times (-1) = 1$. Hence, -1 is the reciprocal of -1.
$j \times (-j) = 1$. Hence, $-j$ is the reciprocal of j.
$(-j) \times j = 1$. Hence, j is the reciprocal of $-j$.

[pages 244-245]

(f)
$$\begin{aligned} j \times x &= 1 \\ (-j) \times (j \times x) &= (-j) \times 1 \\ \big((-j) \times j\big) \times x &= (-j) \times 1 \\ 1 \times x &= -j \\ x &= -j \end{aligned}$$

The truth set is $\{-j\}$.

(g) Similarly, the truth set is $\{-1\}$. Multiply by j, since j is the reciprocal of $-j$.

(h) The truth set is $\{1\}$. Since $j^2 = -1$, multiply by (-1) which is the reciprocal of (-1).

(i) The truth set is $\{-j\}$. Since $j^3 = j^2 \times j = (-1) \times j = -j$, multiply by j, which is the reciprocal of $(-j)$.

Chapter 9

Suggested Test Items

1. Simplify

(a) $-3 - 7$

(b) $9 - 6 - 1$

(c) $-\frac{18}{6}$

(d) $\frac{8}{9} - \frac{2}{3}$

(e) $17 - 2\cdot\frac{1}{3} + \frac{2}{3}$

(f) $\frac{2m}{3}\cdot\frac{-3}{m}$

(g) $\frac{3(x + 1)}{x - 1}\cdot\frac{1}{9(x + 1)}$

(h) $\frac{|3|}{a}\cdot\frac{a}{|-3|}$

(i) $\dfrac{2a}{\frac{1}{3}a}$

(j) $(6x - 2) - (3x + 1)$

(k) $2(m - 1) - (m - 1)$

(ℓ) $\frac{3y - 3}{y - 1}$

(m) $\frac{(k - 1) - (k + 2)}{k - 1}$

(n) $\dfrac{\frac{x + 2}{3}}{\frac{x + 2}{2}}$

(o) $3y - 7x + 5 + 2x - y - x$

(p) $\frac{2x^2y}{z} \cdot \frac{6z^2}{-7xy}$

(q) $(7x^2 - 3x + 2) - (3x^2 - x + 1)$

(r) $\frac{3x - y}{2y - 6x}$

(s) $\frac{x + 1}{3} + \frac{x - 1}{4}$

(t) $\frac{x}{x + \frac{1}{x}}$

(u) $\frac{\frac{2}{3} + \frac{1}{3}}{\frac{2}{3} - \frac{1}{3}}$

(v) $\frac{m}{n} \cdot \frac{n}{m}$

(w) $\frac{m}{n} + \frac{n}{m}$

2. If $x_1 = 6$ and $x_2 = -3$ find

(a) $x_1 - x_2$

(b) $x_2 - x_1$

(c) $|x_1 - x_2|$

(d) $|x_2 - x_1|$

3. Solve

(a) $\frac{x}{2} + \frac{x}{3} = 10$

(b) $\frac{x}{8} = \frac{x - 1}{4}$

(c) $\frac{1}{2} - \frac{1}{7}x = 3x$

(d) $\frac{1}{x} = \frac{3}{7}$

(e) $\frac{1}{3}y + \frac{2}{3} = \frac{1}{7}y$

(f) $\frac{x}{4} - \frac{6}{7} = \frac{5}{8}$

4. Find the truth sets of the following

(a) $y - 3 = 7 - y$

(b) $17 + x > -3 - x$

(c) $\frac{1}{3}a + \frac{1}{6} = \frac{3}{4}a$

(d) $3|m| \leq -|m| + 12$

(e) $\frac{15}{|x|} > 10$

(f) $\frac{y}{4} + \frac{y}{7} = 1$

5. For what numbers is each of the following true?

(a) $0 \cdot x = 0$
(b) $0 = 3x$
(c) $x \cdot 0 = 3$
(d) $\frac{0}{x} = 0$
(e) $|x - 1| < 0$
(f) $3x < 0$

6. If $a > b$ decide if possible whether the following are positive or negative

(a) $a - b$
(b) $\frac{1}{b - a}$
(c) $(a - b)^2$
(d) $|b - a|$
(e) a and b if $ab < o$
(f) a and b if $\frac{a}{b} < o$
(g) $a^2 - ab$
(h) $b^2 - ab$

7. What number must be added to $3a - 2b + c$ to get $a - 6b - 3c$?

8. By what number must $\frac{3}{bc}$ be multiplied to get $17a$?

9. 12 is 30% of what number?

10. If the numerator of the fraction $\frac{5}{12}$ is increased by x where x is positive, the difference in the values of the fractions is $\frac{1}{6}$. Find x.

11. The ratio of antifreeze to water in the radiator of Jim's car is $\frac{3}{7}$. If there are 12 quarts of the mixture, how many quarts of antifreeze are in the mixture?

12. Bob is 4 years older than Don. The sum of their ages is less than 24. Two years from now the sum of their ages will be greater than 20. How old is each boy?